On reductions of families of crystalline Galois representations

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Abstract

Let K be any finite unramified extension of $\mathbb{Q}_p$. We construct analytic families of étale (φ, Γ_K)-modules which correspond to all the effective crystalline characters and some families of n-dimensional crystalline Galois representations of $G_K = \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$. As an application, we compute semisimplified modulo p reductions for some of these families.

Contents

1 Introduction

Throughout this paper p will be a fixed prime number, K a finite unramified extension of $\mathbb{Q}_p$ and E a finite, large enough extension of K with maximal ideal m_E and residue field k_E. We denote f the degree of K over $\mathbb{Q}_p$ and σ the absolute Frobenius of K. We fix once and for all embeddings $K \overset{\varepsilon}{\hookrightarrow} E \hookrightarrow \bar{\mathbb{Q}}_p$ and we let $\tau_j = \varepsilon \circ \sigma^j$ for $j = 0, 1, ..., f-1$. We fix the f-tuple of embeddings $\mid \tau \mid := (\tau_0, \tau_1, ..., \tau_{f-1})$. The map $\xi : E \otimes K \to \prod_{\tau: K \hookrightarrow E} E$ with $\xi(x \otimes y) = (x\tau(y))_\tau$ is a ring isomorphism. We denote $E^{|\tau|} = \prod_{\tau: K \hookrightarrow E} E$. The ring automorphism $1_E \otimes \sigma : E \otimes K \to E \otimes K$ transforms via ξ to the automorphism $\varphi : E^{|\tau|} \to E^{|\tau|}$ with $\varphi(x_0, x_1, ..., x_{f-1}) = (x_1, ..., x_{f-1}, x_0)$. We denote $e_j = (0, ..., 1_j, ..., 0)$ the idempotent of $E^{|\tau|}$ where the 1 occurs in the τ_j-th coordinate, for each $j \in \{0, 1, ..., f-1\}$. Let $\rho : G_K \to GL_E(V)$ be a continuous E-linear representation of G_K =Gal$(\bar{\mathbb{Q}}_p/K)$. Recall ([6], §3) that $D_{cris}(V) = (\mathbb{B}_{cris} \otimes_{\mathbb{Q}_p} V)^{G_K}$, where $\mathbb{B}_{cris}$ is the ring constructed by Fontaine in [13], is a filtered φ-module over K with E-coefficients and V is crystalline if and only if $D_{cris}(V)$ is free over $E \otimes K$ of rank $\dim_E V$.

Throughout the paper we assume that V is crystalline. One can easily prove (c.f. [17] appendix B) that V is crystalline as an E-linear representation of G_K if and only if it is crystalline as a $\mathbb{Q}_p$-linear representation of G_K. We may therefore extend E whenever appropriate without affecting crystallinity. By a variant of the fundamental theorem of Colmez and Fontaine (c.f. [10], Théorème A) for non-trivial coefficients, the functor $V \mapsto D_{cris}(V)$ is an equivalence of categories from the category of crystalline E-linear representations of G_K to the category of weakly admissible filtered φ-modules (D, φ) over K with E-coefficients (see [6], §3). Such a filtered module D is a module over $E \otimes K$ and may be viewed as a module over $E^{|\tau|}$ via the ring isomorphism ξ defined above. Its Frobenius endomorphism is bijective and semilinear with respect to the automorphism φ of $E^{|\tau|}$. We filter each component $D_i = e_i D$ of D by setting $\mathrm{Fil}^j D_i = e_i \mathrm{Fil}^j D$. An integer j is called a labeled Hodge-Tate weight with respect to the embedding τ_i of K in E if and only if $e_i \mathrm{Fil}^{-j} D \neq e_i \mathrm{Fil}^{-j+1} D$, and is counted with multiplicity $\dim_E \left(e_i \mathrm{Fil}^{-j} D / e_i \mathrm{Fil}^{-j+1} D\right)$. Since the Frobenius endomorphism of D restricts to an E-linear isomorphism from D_i to D_{i-1} for all i, the components D_i are equidimensional over E. As a consequence, there are n =rank$_{E \otimes K}(D)$ labeled Hodge-Tate weights for each embedding, counting multiplicities.

The labeled Hodge-Tate weights of D are by definition the f-tuple of multisets $(W_i)_{\tau_i}$, where each such multiset W_i contains n integers, the opposites of the jumps of the filtration of D_i. The characteristic polynomial of a crystalline E-linear representation of G_K is the characteristic polynomial of the $E^{|\tau|}$-linear map φ^f, where (D, φ) is the weakly admissible filtered φ-module corresponding to it by the Colmez-Fontaine theorem. A filtered φ-module (D, φ) is called F-semisimple, non-F-semisimple, or F-scalar if the $E^{|\tau|}$-linear map φ^f has the corresponding property.

We may twist D by some appropriate rank one weakly admissible filtered φ-module (see Proposition 2.4) and assume that $W_i = \{-w_{in-1} \leq ... \leq -w_{i2} \leq -w_{i1} \leq 0\}$ for all $i = 0, 1, ..., f-1$. The Hodge-Tate weights of a crystalline representation V are the opposites of the jumps of the filtration of $D_{cris}(V)$. If they are all non positive the crystalline representation is called effective.

1.1 Étale (φ, Γ)-modules and Wach modules

Let $K_n = K(\mu_{p^n})$ where μ_{p^n} is a primitive p^n-th root of unity inside $\bar{\mathbb{Q}}_p$ and $K_\infty = \cup_{n\geq 1} K_n$. Let $\chi : G_K \to \mathbb{Z}_p^\times$ be the cyclotomic character. We denote $H_K = \ker \chi =$Gal$(\bar{\mathbb{Q}}_p/K_\infty)$ and $\Gamma_K = G_K/H_K =$Gal(K_∞/K). Let $\mathbb{A}_K$ be the ring defined by $\mathbb{A}_K = \{\sum_{n=-\infty}^{\infty} \alpha_n \pi_K^n : \alpha_n \in \mathcal{O}_K$ and $\lim_{n\to -\infty} \alpha_n = 0\}$ where π_K is a formal variable. $\mathbb{A}_K$ is equipped with a Frobenius endomorphism φ which extends the absolute Frobenius of $\mathcal{O}_K$ and is such that $\varphi(\pi_K) = (1+\pi_K)^p - 1$. It is also equipped with a commuting with the Frobenius Γ_K-action which is $\mathcal{O}_K$-linear and is such that $\gamma(\pi_K) = (1+\pi_K)^{\chi(\gamma)} - 1$ for all $\gamma \in \Gamma_K$. For simplicity we write π instead of π_K. The ring $\mathbb{A}_K$ is local with maximal ideal (p), residue field $\mathbb{E}_K = k_K((\pi))$, where k_K is the residue field of K, and fraction field $\mathbb{B}_K = \mathbb{A}_K[\frac{1}{p}]$. A (φ, Γ)-module over $\mathbb{A}_K$ (respectively $\mathbb{B}_K$) is a free $\mathbb{A}_K$-module of finite type (respectively finite dimensional $\mathbb{B}_K$-vector space) with continuous commuting semilinear actions of φ and Γ_K. A (φ, Γ)-module M over $\mathbb{A}_K$ is called étale if $\varphi^*(M) = M$, where $\varphi^*(M)$ is the $\mathbb{A}_K$-module generated by the set $\varphi(M)$.

A (φ, Γ)-module M over $\mathbb{B}_K$ is called étale if it contains an $\mathbb{A}_K$-lattice which is étale over $\mathbb{A}_K$. Fontaine proved that there is an equivalence of categories between p-adic (respectively $\mathbb{Z}_p$-adic) representations of G_K and étale (φ, Γ)-modules over $\mathbb{B}_K$ (respectively $\mathbb{A}_K$) given by the functor $V \mapsto D(V) = (\mathbb{B} \otimes_{\mathbb{Q}_p} V)^{H_K}$ (respectively $T \mapsto D(T) = (\mathbb{A} \otimes_{\mathbb{Z}_p} T)^{H_K}$), where $\mathbb{A}$ and $\mathbb{B}$ are rings constructed by Fontaine (c.f. [14]) with the properties that $\mathbb{A}_K = \mathbb{A}^{H_K}$ and $\mathbb{B}_K = \mathbb{B}^{H_K}$. When studying the category of E (respectively $\mathcal{O}_E$)-linear representations of G_K we replace $\mathbb{B}_K$ by $E \otimes_{\mathbb{Q}_p} \mathbb{B}_K$ (respectively $\mathbb{A}_K$ by $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_K$). In this case the φ and Γ_K-actions are E (respectively $\mathcal{O}_E$)-linear. There is a ring isomorphism $\xi : \mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_K^+ \to \prod_{\tau : K \hookrightarrow E} \mathcal{O}_E[[\pi]]$ given by $\xi(a \otimes b) = (a\tau_0(b), a\tau_1(b), ..., a\tau_{f-1}(b))$, where $\tau_i\left(\sum_{n=0}^{\infty} \beta_n \pi^n\right) = \sum_{n=0}^{\infty} \tau_i(\beta_n)\pi^n$ for all $b = \sum_{n=0}^{\infty} \beta_n \pi^n \in \mathbb{A}_K^+$. The ring $\mathcal{O}_E[[\pi]]^{|\tau|} := \prod_{\tau : K \hookrightarrow E} \mathcal{O}_E[[\pi]]$ is equipped with $\mathcal{O}_E$-linear actions of φ and Γ_K given by $\varphi(\alpha_0(\pi), \alpha_1(\pi), ..., \alpha_{f-1}(\pi)) = (\alpha_1(\varphi(\pi)), ..., \alpha_{f-1}(\varphi(\pi)), \alpha_0(\varphi(\pi)))$ and $\gamma(\alpha_0(\pi), \alpha_1(\pi), ..., \alpha_{f-1}(\pi)) = (\alpha_0(\gamma\pi), \alpha_1(\gamma\pi), ..., \alpha_{f-1}(\gamma\pi))$ for all $\gamma \in \Gamma_K$. A natural question is to determine the types of étale (φ, Γ)-modules which correspond to crystalline representations via Fontaine's functor. An answer is given by the following Theorem of Berger who used previous work of Wach [19], [20] and Colmez [8]. In the following Theorem and throughout the paper, $\mathbb{A}_K^+ = \mathcal{O}_K[[\pi]] \subset \mathbb{A}_K$ and $\mathbb{B}_K^+ = \mathbb{A}_K^+[\frac{1}{p}] \subset \mathbb{B}_K$.

Theorem 1.1 *([1] II.1, III.4) Let V be an E-linear representation of G_K. The representation V is crystalline with Hodge-Tate weights in $[-k, 0]$, for some non negative integer k, if and only if there exists an $E \otimes_{\mathbb{Q}_p} \mathbb{B}_K^+$-module $N(V)$ contained in $D(V)$ such that:*

1. $N(V)$ is free of rank $d = \dim_E(V)$ over $E \otimes_{\mathbb{Q}_p} \mathbb{B}_K^+$;

2. The Γ_K-action preserves $N(V)$ and is trivial on $N(V)/\pi N(V)$;

3. $\varphi(N(V)) \subset N(V)$ *and* $N(V)/\varphi^*(N(V))$ *is killed by* q^k*, where* $q = \varphi(\pi)/\pi$*. The module* $N(V)$ *is endowed with the filtration* $Fil^j(N(V)) = \{x \in N(V) : \varphi(x) \in q^j N(V)\}$ *for* $j \geq 0$ *and* $N(V)/\pi N(V)$ *is endowed with the induced filtration. Then*

$$D_{cris}(V) \simeq (N(V)/\pi N(V))$$

as filtered φ*-modules over* $E^{|\tau|}$*. Moreover, if* T *is a* G_K*-stable,* $\mathcal{O}_E$*-lattice in* V*, then* $N(T) = D(T) \cap N(V)$ *is a* $\mathcal{O}_E[[\pi]]^{|\tau|}$*-lattice in* $N(V)$ *and the functor* $T \mapsto N(T)$ *gives a bijection between the* G_K*-stable,* $\mathcal{O}_E$*-lattices in* V *and the* $\mathcal{O}_E[[\pi]]^{|\tau|}$*-lattices in* $N(V)$ *satisfying the following conditions:*
1. $N(T)$ *is free of rank* $d = \dim_E(V)$ *over* $\mathcal{O}_E[[\pi]]^{|\tau|}$*;*
2. The Γ_K*-action preserves* $N(T)$ *and is trivial on* $N(T)/\pi N(T)$*;*
3. $\varphi(N(T)) \subset N(T)$ *and* $N(T)/\varphi^*(N(T))$ *is killed by* q^k*.* $\square$

Such modules $N(V)$ are called Wach modules over $E \otimes_{\mathbb{Q}_p} \mathbb{B}^+_K$ (respectively $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}^+_K$). The étale (φ, Γ_K)-module $D(V)$ is obtained from $N(V)$ by extension of scalars. Constructing the Wach module of a crystalline representation amounts to explicitly constructing the crystalline representation. It has the advantage that instead of working with the more complicated rings $\mathbb{A}_K$ and $\mathbb{B}_K$, one works with the simpler ones $\mathbb{A}^+_K$ and $\mathbb{B}^+_K$.

The purpose of this article is to construct families of Wach modules corresponding to effective, crystalline E-linear representations of G_K, and use them to study the reductions of the corresponding crystalline representations modulo p.

1.2 Overview of the article

In section 2 we construct the Wach modules corresponding to all effective, crystalline $E^\times$-valued characters of G_K. In Section 3 we provide a general method for constructing families of Wach modules of two-dimensional crystalline representations and, as an application, we prove that all the members of any such family have the same semisimplified modulo p reduction (Theorem 3.7). Our method generalizes the method used by Berger-Li-Zhu in the two-dimensional case, when $K = \mathbb{Q}_p$. In Sections 4 and 6 we apply this method to construct some families of Wach modules of two-dimensional crystalline representations. The semisimplified mod p reductions of some of these families are computed in Sections 5 and 6. In Section 5, we also obtain explicit formulas for the semisimplified modulo p reductions of any reducible two-dimensional E-representations of G_K (Theorem 5.6). For this, a classification of F-semisimple, two-dimensional crysrtalline representations of G_K with arbitrary, large enough, coefficients is essential. Such a classification is recalled in Section 5.2. For the proofs of the corresponding results see [12]. In Section 7 we explain how to use the method of Section 3 to construct families of Wach modules of n-dimensional representation of G_K. Moreover, we construct a family of three-dimensional crystalline representations of $G_{\mathbb{Q}_p}$ and compute its reduction modulo p.

Notation: *Assume that after ordering the weights* k_i *and omitting possibly repeated weights we get* $w_0 < w_1 < ... < w_{t-1}$*, where* w_0 is the smallest weight ,..., w_{t-1} the largest weight, with $1 \leq t \leq f$. *For convenience let* $w_{-1} = 0$*. Let* $I_0 = \{0, 1, ..., f-1\}$*,* $I_0^+ = \{i \in I_0 : k_i > 0\}$*,* $I_1 = \{i \in I_0 : k_i > w_0\}$*,* $I_2 = \{i \in I_0 : k_i > w_1\}, ..., I_{t-1} = \{i \in I_0 : k_i > w_{t-2}\}$ $=\{i \in I_0 : k_i = w_{t-1}\}$ *and* $I_t = \varnothing$*. We denote* $k = w_{t-1} = \max\{k_0, k_1, ..., k_{f-1}\}$ *and* $\vec{w} = (k_1, k_2, ..., k_{f-1}, k_0)$*. For each* $J \subset I_0$ *we write* $f_J = \sum_{i \in J} e_i$ *and* $E^{|\tau_J|} = f_J \cdot E^{|\tau|}$*. If* $\vec{x} \in E^{|\tau|}$ *we denote* x_i *its i-th coordinate,* $Nm_\varphi(\vec{x}) := \prod_{i=0}^{f-1} \varphi^i(\vec{x})$*,*

$v_p(Nm_\varphi(\vec{x})) := v_p\left(\prod_{i=0}^{f-1} x_i\right)$ *(where v_p is the normalized p-adic valuation of $\bar{\mathbb{Q}}_p$) and $J_{\vec{x}}$ the set $\{i \in I_0 : x_i \neq 0\}$. If ℓ is an integer, we write $\vec{\ell} = (\ell, \ell, ..., \ell)$ and $v_p(\vec{x}) > \vec{\ell}$ (respectively $v_p(\vec{x}) \geq \vec{\ell}$) if and only if $v_p(x_i) > \ell$ (respectively $v_p(x_i) \geq \ell$) for all i. For any matrix $A \in M_2(E^{|\tau|})$, we write $Nm_\varphi(A) := A\varphi(A)...\varphi^{f-1}(A)$ with φ acting on each entry of A.*

2 Construction of the effective rank-one Wach modules

In this section we construct the rank one Wach modules over $\mathcal{O}_E[[\pi]]^{|\tau|}$ with labeled Hodge-Tate weights $\{-k_i\}_{\tau_i}$.

Definition 2.1 *Let $q = \frac{\varphi(\pi)}{\pi}$ where $\varphi(\pi) = (1+\pi)^p - 1$. We define $q_1 = q$, $q_n = \varphi^{n-1}(q)$ and $\lambda_f = \prod_{n=0}^{\infty}\left(\frac{q_{nf+1}}{p}\right)$. For each $\gamma \in \Gamma_K$, we define $\lambda_{f,\gamma} = \frac{\lambda_f}{\gamma\lambda_f}$.*

Lemma 2.2 *For each $\gamma \in \Gamma_K$, the functions $\lambda_f,\ \lambda_{f,\gamma} \in \mathbb{Q}_p[[\pi]]$ have the following properties:* (1) $\lambda_f(0) = 1$; (2) $\lambda_{f,\gamma} \in 1 + \pi\mathbb{Z}_p[[\pi]]$.

Proof. (1) This is clear since $\frac{q_n(0)}{p} = 1$ for all $n \geq 1$. (2) One can easily check that $\frac{q}{\gamma q} \in 1+\pi\mathbb{Z}_p[[\pi]]$. From this we deduce that $\lambda_{f,\gamma} \in 1 + \pi\mathbb{Z}_p[[\pi]]$. ■

Consider the rank one module $\boldsymbol{N}_{C,\vec{w}} = \left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta$ equipped with semilinear actions of φ and Γ_K defined by $\varphi(\eta) = (Cq^{k_1}, q^{k_2}, ..., q^{k_{f-1}}, q^{k_0})\eta$ and $\gamma(\eta) = (g_1^\gamma(\pi), g_2^\gamma(\pi), ...g_{f-1}^\gamma(\pi), g_0^\gamma(\pi))\eta$ for all $\gamma \in \Gamma_K$, where $C \in \mathcal{O}_E^\times$. We define the functions $g_i(\pi) = g_i^\gamma(\pi) \in \mathcal{O}_E[[\pi]]$ appropriately to make $\boldsymbol{N}_{C,\vec{w}}$ a Wach module over $\mathcal{O}_E[[\pi]]^{|\tau|}$. The actions of φ and γ should commute and a short computation shows that g_0 should satisfy the equation

$$\varphi^f(g_0) = g_0(\frac{\gamma q}{q})^{k_0}\left(\varphi(\frac{\gamma q}{q})\right)^{k_1} ... \left(\varphi^{f-1}(\frac{\gamma q}{q})\right)^{k_{f-1}}.$$

Lemma 2.3 *The equation $\varphi^f(g_0) = g_0(\frac{\gamma q}{q})^{k_0}\left(\varphi(\frac{\gamma q}{q})\right)^{k_1} ... \left(\varphi^{f-1}(\frac{\gamma q}{q})\right)^{k_{f-1}}$ has a unique $\equiv 1 \bmod \pi$ solution in $\mathbb{Z}_p[[\pi]]$ given by $g_0 = (\lambda_{f,\gamma})^{k_0}(\varphi(\lambda_{f,\gamma}))^{k_1}\left(\varphi^2(\lambda_{f,\gamma})\right)^{k_2} ... \left(\varphi^{f-1}(\lambda_{f,\gamma})\right)^{k_{f-1}}$.*

Proof. Notice that $\varphi^f(\lambda_f) = \frac{\lambda_f}{(\frac{q}{p})}$ and $\varphi^f(\gamma\lambda_f) = \frac{\gamma\lambda_f}{(\frac{\gamma q}{p})}$, hence $\lambda_{f,\gamma} = \frac{\lambda_f}{\gamma\lambda_f}$ solves the equation $\varphi^f(u) = u\left(\frac{\gamma q}{q}\right)$. Since the actions of φ and γ on $\mathbb{Z}_p[[\pi]]$ commute,

$$g_0 = (\lambda_{f,\gamma})^{k_0}(\varphi(\lambda_{f,\gamma}))^{k_1}\left(\varphi^2(\lambda_{f,\gamma})\right)^{k_2} ... \left(\varphi^{f-1}(\lambda_{f,\gamma})\right)^{k_{f-1}}$$

is a solution of the equation above and $g_0 \equiv 1 \bmod \pi$ by Lemma 2.2. If u_1, u_2 are two solutions of

$$\varphi^f(u) = u(\frac{\gamma q}{q})^{k_0}(\varphi(\frac{\gamma q}{q}))^{k_1}...(\varphi^{f-1}(\frac{\gamma q}{q}))^{k_{f-1}}$$

congruent to $1 \bmod \pi$, then $(\frac{u_1}{u_2}) \in \mathbb{Z}_p[[\pi]]$ is fixed by φ^f and is congruent to $1 \bmod \pi$, hence equals 1. ■

Commutativity of φ with the Γ_K-action implies that
$g_1 = (\frac{q}{\gamma q})^{k_1} \left(\varphi(\frac{q}{\gamma q})\right)^{k_2} ... \left(\varphi^{f-2}(\frac{q}{\gamma q})\right)^{k_{f-1}} \left(\varphi^{f-1}(\lambda_{f,\gamma})\right)^{k_0} \left(\varphi^{f}(\lambda_{f,\gamma})\right)^{k_1} ... \left(\varphi^{2f-2}(\lambda_{f,\gamma})\right)^{k_{f-1}}$
....
$g_{f-2} = (\frac{q}{\gamma q})^{k_{f-2}} \left(\varphi(\frac{q}{\gamma q})\right)^{k_{f-1}} \left(\varphi^{2}(\lambda_{f,\gamma})\right)^{k_0} \left(\varphi^{3}(\lambda_{f,\gamma})\right)^{k_1} ... \left(\varphi^{f+1}(\lambda_{f,\gamma})\right)^{k_{f-1}}$,
$g_{f-1} = (\frac{q}{\gamma q})^{k_{f-1}} \left(\varphi(\lambda_{f,\gamma})\right)^{k_0} \left(\varphi^{2}(\lambda_{f,\gamma})\right)^{k_1} \left(\varphi^{3}(\lambda_{f,\gamma})\right)^{k_2} ... \left(\varphi^{f}(\lambda_{f,\gamma})\right)^{k_{f-1}}$.
From the equations above and Lemma 2.2 we deduce that $g_i \equiv 1 \bmod \pi$ for all i.

Proposition 2.4 *Let $\boldsymbol{N}_{C,\vec{w}} = \left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta$ be equipped with semilinear φ and Γ_K-actions defined by $\varphi(\eta) = (Cq^{k_1}, q^{k_2}, ..., q^{k_{f-1}}, q^{k_0})\eta$ and $\gamma(\eta) = (g_1^{\gamma}(\pi), g_2^{\gamma}(\pi), ...g_{f-1}^{\gamma}(\pi), g_0^{\gamma}(\pi))\eta$ for the $g_i(\pi) = g_i^{\gamma}(\pi)$ defined above. The module $\boldsymbol{N}_{C,\vec{w}}$ is a Wach module over $\mathcal{O}_E[[\pi]]^{|\tau|}$ with labeled Hodge-Tate weights $\{-k_i\}_{\tau_i}$, and is such that*

$$D_{C,\vec{w}} \simeq E^{|\tau|} \bigotimes_{\mathcal{O}_E^{|\tau|}} (\boldsymbol{N}_{C,\vec{w}}/\pi\boldsymbol{N}_{C,\vec{w}})$$

as filtered φ-modules. The filtered φ-module $D_{C,\vec{w}} = \left(E^{|\tau|}\right)\eta$ has Frobenius endomorphism $\varphi(\eta) = (Cp^{k_1}, p^{k_2}, ..., p^{k_{f-1}}, p^{k_0})\eta$ and filtration

$$Fi\ell^j(D_{C,\vec{w}}) = \begin{cases} E^{|\tau_{I_0}|}\eta & \text{if } j \leq w_0, \\ E^{|\tau_{I_1}|}\eta & \text{if } 1+w_0 \leq j \leq w_1, \\ \text{...} & \\ E^{|\tau_{I_{t-1}}|}\eta & \text{if } 1+w_{t-2} \leq j \leq w_{t-1}, \\ 0 & \text{if } j \geq 1+w_{t-1}. \end{cases}$$

Proof. (a) To prove that Γ_K acts on $\boldsymbol{N}_{C,\vec{w}}$, it suffices to prove that $g_i^{\gamma_1\gamma_2}(\pi) = g_i^{\gamma_1}\gamma_1(g_i^{\gamma_2})$ for all $\gamma_1, \gamma_2 \in \Gamma_K$ and $i \in I_0$. This follows immediately from the cocycle relations $\frac{q}{\gamma_1\gamma_2(q)} = \frac{q}{\gamma_1(q)}\gamma_1\left(\frac{q}{\gamma_2(q)}\right)$ and $\lambda_{f,\gamma_1\gamma_2} = \lambda_{f,\gamma_1}\gamma_1(\lambda_{f,\gamma_2})$ along with the definition of the $g_i^{\gamma}(\pi)$, $i \in I_0$. Since $g_i^{\gamma}(\pi) \equiv 1 \bmod \pi$ for all $i \in I_0$, Γ_K acts trivially on $\boldsymbol{N}_{C,\vec{w}}/\pi\boldsymbol{N}_{C,\vec{w}}$.
(b) Let $k = \max\{k_0, k_1, ..., k_{f-1}\}$ and $\varphi^*(\boldsymbol{N}_{C,\vec{w}})$ be the $\mathcal{O}_E[[\pi]]^{|\tau|}$-linear span of the set $\varphi(\boldsymbol{N}_{C,\vec{w}})$. Since $q^k\eta = \sum_{i=0}^{f-1}(q^{k-k_i}C_ie_i)\ \varphi(\eta) \in \varphi^*(\boldsymbol{N}_{C,\vec{w}})$, where $C_1 = C^{-1}$ and $C_i = 1$ if $i \neq 1$, q^k kills $\boldsymbol{N}_{C,\vec{w}}/\varphi^*(\boldsymbol{N}_{C,\vec{w}})$.
(c) To compute the filtration of $\boldsymbol{N}_{C,\vec{w}}$, we use the fact that $q^j \mid \varphi(x)$ if and only if $\pi^j \mid x$ for any $x \in \mathcal{O}_E[[\pi]]$. Let $x = (x_0, x_1, ..., x_{f-1})\eta \in \boldsymbol{N}_{C,\vec{w}}$. By Theorem 1.1, $x \in \mathrm{Fil}^j\boldsymbol{N}_{C,\vec{w}}$ if and only if $\varphi(x) \in q^j\boldsymbol{N}_{C,\vec{w}}$ or equivalently $q^j \mid \varphi(x_i)q^{k_i}$ for all $i \in I_0$. If $j \leq k_i$ there are no restrictions on x_i and if $j > k_i$, then this is equivalent to $x_i \equiv 0 \bmod \pi^{j-k_i}$. Therefore,

$$e_i\mathrm{Fil}^j\boldsymbol{N}_{C,\vec{w}} = \begin{cases} e_i\boldsymbol{N}_{C,\vec{w}} \text{ if } j \leq k_i, \\ e_i\pi^{j-k_i}\mathcal{O}_E[[\pi]]\eta \text{ if } j \geq 1+k_i. \end{cases}$$

This implies that

$$E^{|\tau|} \bigotimes_{\mathcal{O}_E^{|\tau|}} e_i\mathrm{Fil}^j\left(\boldsymbol{N}_{C,\vec{w}}/\pi\boldsymbol{N}_{C,\vec{w}}\right) = \begin{cases} e_iE^{|\tau|}\overline{\eta} \text{ if } j \leq k_i, \\ 0 \text{ if } j \geq k_i. \end{cases}$$

For the filtration, notice that

$$E^{|\tau|} \bigotimes_{\mathcal{O}_E^{|\tau|}} \mathrm{Fil}^j \left(\boldsymbol{N}_{C,\vec{w}}/\pi \boldsymbol{N}_{C,\vec{w}}\right) = \bigoplus_{i=0}^{f-1} \left(E^{|\tau|} \bigotimes_{\mathcal{O}_E^{|\tau|}} e_i \mathrm{Fil}^j (\boldsymbol{N}_{C,\vec{w}}/\pi \boldsymbol{N}_{C,\vec{w}}) \right)$$

(c.f.[12, § 2.3]). The isomorphism of filtered modules is obvious. ■

Proposition 2.5 *Let $k_0, k_1, ..., k_{f-1}$ be integers, not necessarily non negative. The weakly admissible rank one filtered φ-modules over $E^{|\tau|}$ with labeled Hodge-Tate weights $\{-k_i\}_{\tau_i}$ are of the form $D_{\vec{w},\vec{\alpha}} = E^{|\tau|}\eta$, with $\varphi(e) = (\alpha_0, \alpha_1, ..., \alpha_{f-1})\eta$ for some $\vec{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_{f-1}) \in (E^\times)^{|\tau|}$, such that $v_p(Nm_\varphi(\vec{\alpha})) = \sum_{i \in I_0} k_i$ and*

$$Fil^j(D_{\vec{w},\vec{\alpha}}) = \begin{cases} E^{|\tau_{I_0}|}\eta & \text{if } j \leq w_0, \\ E^{|\tau_{I_1}|}\eta & \text{if } \ 1 + w_0 \leq j \leq w_1, \\ \dots\dots\dots\dots\dots\dots & \\ E^{|\tau_{I_{t-1}}|}\eta & \text{if } 1 + w_{t-2} \leq j \leq w_{t-1}, \\ \quad 0 & \text{if } j \geq 1 + w_{t-1}. \end{cases}$$

The filtered φ-modules $D_{\vec{w},\vec{\alpha}}$ and $D_{\vec{v},\vec{\beta}}$ are isomorphic if and only if $\vec{w} = \vec{v}$ and $Nm_\varphi(\vec{\alpha}) = Nm_\varphi(\vec{\beta})$.

Proof. Mimic the proofs in [12], § 2 and 3. ■

Corollary 2.6 *All effective crystalline $E^\times$-valued characters of G_K are those constructed in Proposition 2.4.*

3 General construction of families of effective Wach modules of arbitrary rank

We extend the method used by Berger-Li-Zhu (c.f.[4]) to the case of G_K, where K is any finite, unramified extension of $\mathbb{Q}_p$. In order to construct the Wach module of an effective crystalline representation, we need to exhibit matrices Π and G_γ such that $\Pi\varphi(G_\gamma) = G_\gamma\gamma(\Pi)$ for all $\gamma \in \Gamma_K$, with the additional properties imposed by Theorem 1.1. In the two-dimensional case, when $K = \mathbb{Q}_p$, assuming that the valuation of some parameter is suitably large, it is trivial to write down such a matrix Π and the main difficulty is to construct the commuting with it Γ_K-action. When $K \neq \mathbb{Q}_p$, even in the two-dimensional case, finding a matrix Π which gives rise to a prescribed weakly admissible filtration seems to be at least as hard as constructing the Γ_K-action. Assuming that such a matrix Π is available, it is usually very hard to explicitly write down the matrices G_γ, with the split-reducible case being an exception. Instead, we prove that such matrices exist using a successive approximation argument.

Let $\mathcal{S} = \{X_i \ ; \ i = 0, 1, ..., m-1\}$ be any set of indeterminates. We extend the actions of φ and Γ_K on the ring $\mathcal{O}_E[[\pi]]^{|\tau|} := \prod_{\tau: K \hookrightarrow E} \mathcal{O}_E[[\pi]]$ to an action on $\mathcal{O}_E[[\pi, \mathcal{S}]]^{|\tau|}$ by letting φ and Γ_K act trivially on each indeterminate X_i. We let φ and Γ_K act on the matrices of $\mathcal{M}_n := M_n(\mathcal{O}_E[[\pi, \mathcal{S}]]^{|\tau|})$ entry-wise, for any integer $n \geq 2$. For any integer $s \geq 0$, we write $\vec{\pi}^s = (\pi^s, \pi^s, ..., \pi^s)$, and for

any $\alpha \in \mathcal{O}_E[[\pi, \mathcal{S}]]$ and any vector $\vec{r} = (r_0, r_1, ..., r_{f-1})$ with non negative integer coordinates we write $\alpha^{\vec{r}} = (\alpha^{r_0}, \alpha^{r_1}, ..., \alpha^{r_{f-1}})$. As usual, we let k_i be non negative integers we denote $k = \max\{k_0, k_1, ..., k_{f-1}\}$ and we let ℓ be any integer with $\ell \geq k$.

Lemma 3.1 *Let $\Pi_i = \Pi_i(\mathcal{S}),\ i = 0, 1, ..., f-1$ be matrices in $M_n(\mathcal{O}_E[[\pi, \mathcal{S}]])$ such that $\det(\Pi_i) = C_i q^{k_i}$, with $C_i \in \mathcal{O}_E^{\times}$. We denote $\Pi(\vec{\mathcal{S}})$ the matrix $\Pi_1 \times \Pi_2 \times ... \times \Pi_{f-1} \times \Pi_0$ and view it as an element of $\mathcal{M}_n$ in a natural way. We denote $P_i = P_i(\mathcal{S})$ the reduction of $\Pi_i \bmod \pi$ for all i. We assume that, for each $\gamma \in \Gamma_K$, there exists a matrix $G_\gamma^{(\ell)} = G_\gamma^{(\ell)}(\vec{X}) \in \mathcal{M}_n$ such that:*
(i) $G_\gamma^{(\ell)}(\vec{X}) \equiv \vec{Id} \bmod \pi$;
(ii) $G_\gamma^{(\ell)}(\vec{\mathcal{S}})\gamma(\Pi(\vec{\mathcal{S}})) - \Pi(\vec{\mathcal{S}})\varphi(G_\gamma^{(\ell)}(\vec{\mathcal{S}})) \in \vec{\pi}^\ell \mathcal{M}_n$;
(iii) There is no nonzero matrix $H \in M_n(\mathcal{O}_E[[\mathcal{S}]]^{|\tau|})$ such that $HN = p^{ft}NH$ for some $t > 0$, with $N = Nm_\varphi(\Pi^{(0)})$, where $\Pi^{(0)} = P_1 \times P_2 \times ... \times P_{f-1} \times P_0$;
(iv) For each $s \geq \ell + 1$, the operator

$$H \longmapsto H - QH(p^{f(s-1)}Q^{-1}) : M_n(\mathcal{O}_E[[\mathcal{S}]]) \longrightarrow M_n(\mathcal{O}_E[[\mathcal{S}]])$$

is surjective, where $Q = P_1 P_2 ... P_{f-1} P_0$. Then for each $\gamma \in \Gamma_K$ there exists a unique matrix $G_\gamma(\vec{\mathcal{S}}) \in \mathcal{M}_n$ such that (i) $G_\gamma(\vec{\mathcal{S}}) \equiv \vec{Id} \bmod \pi$ and (ii) $\Pi(\vec{\mathcal{S}})\varphi(G_\gamma(\vec{\mathcal{S}})) = G_\gamma(\vec{\mathcal{S}})\gamma(\Pi(\vec{\mathcal{S}}))$.

Proof. *Uniqueness:* Suppose that the matrices $G_\gamma(\vec{\mathcal{S}})$ and $G'_\gamma(\vec{\mathcal{S}})$ both satisfy the conclusions of the lemma, and let $H = G'_\gamma(\vec{\mathcal{S}})G_\gamma(\vec{\mathcal{S}})^{-1}$. We easily see that $H \in \vec{Id} + \vec{\pi}\mathcal{M}_n$ and $H\Pi(\vec{\mathcal{S}}) = \Pi(\vec{\mathcal{S}})\varphi(H)$. We'll show that $H = \vec{Id}$. We write $H = \vec{Id} + \pi^t H_t + ...$, where $H_t \in M_n(\mathcal{O}_E[[\mathcal{S}]]^{|\tau|})$, $t \geq 1$ and $\Pi(\vec{\mathcal{S}}) = \Pi^{(0)} + \pi\Pi^{(1)} + ...$, and we will show that $H_t = 0$. Since $H\Pi(\vec{\mathcal{S}}) = \Pi(\vec{\mathcal{S}})\varphi(H)$, we have $(H - \vec{Id})\Pi(\vec{\mathcal{S}}) = \Pi(\vec{\mathcal{S}})\varphi(H - \vec{Id})$. We divide both sides of this equation by π^t (using that $\varphi(\pi) = q\pi$) and reduce mod π. This gives $H_t\Pi^{(0)} = p^t\Pi^{(0)}\varphi(H_t)$ (since $q \equiv p \bmod \pi$), which implies that $H_t N = p^{ft}N\varphi^f(H_t)$, with $N = Nm_\varphi(\Pi^{(0)})$. Since φ acts trivially on X_i and $\mathcal{O}_E$, φ^f acts trivially on $M_n(\mathcal{O}_E[[\mathcal{S}]]^{|\tau|})$, therefore $H_t N = p^{ft}NH_t$ and $H_t = 0$ by assumption (iii) of the lemma. *Existence:* Fix a $\gamma \in \Gamma_K$. By assumption, there a exists a matrix $G_\gamma^{(\ell)} \in \vec{Id} + \vec{\pi}^\ell\ \mathcal{M}_n$ such that $G_\gamma^{(\ell)} - \Pi(\vec{\mathcal{S}})\varphi(G_\gamma^{(\ell)})\gamma(\Pi(\vec{\mathcal{S}})^{-1}) = \vec{\pi}^\ell R^{(\ell)}$ for some $R^{(\ell)} = R^{(\ell)}(\gamma) \in \mathcal{M}_n$. We'll prove that, for each $s \geq \ell + 1$, there exist matrices $R^{(s)} = R^{(s)}(\gamma)$ and $G_\gamma^{(s)} \in \mathcal{M}_n$ such that $G_\gamma^{(s)} \equiv G_\gamma^{(s-1)} \bmod \vec{\pi}^{s-1}\mathcal{M}_n$ and $G_\gamma^{(s)}\gamma(\Pi(\vec{\mathcal{S}})) - \Pi(\vec{\mathcal{S}})\varphi(G_\gamma^{(s)}) = \vec{\pi}^s R^{(s)}$. Let $G_\gamma^{(s)} = G_\gamma^{(s-1)} + \vec{\pi}^{s-1}H^{(s)}$, where $H^{(s)} \in M_2(\mathcal{O}_E[[\mathcal{S}]]^{|\tau|})$ and denote $\bar{R}^{(s)} \equiv R^{(s)} \bmod \pi$. We need

$$\left(G_\gamma^{(s-1)} + \vec{\pi}^{(s-1)}H^{(s)}\right)\gamma(\Pi(\vec{\mathcal{S}})) - \Pi(\vec{\mathcal{S}})\left(\varphi(G_\gamma^{(s-1)}) + \varphi\left(\vec{\pi}\right)^{(s-1)}\varphi\left(H^{(s)}\right)\right) \in \vec{\pi}^s\mathcal{M}_n$$

if and only if

$$G_\gamma^{(s-1)}\gamma(\Pi(\vec{\mathcal{S}})) - \Pi(\vec{\mathcal{S}})\varphi(G_\gamma^{(s-1)}) + \vec{\pi}^{(s-1)}H^{(s)}\gamma(\Pi(\vec{\mathcal{S}})) - \left(\vec{q\pi}\right)^{(s-1)}\Pi(\vec{\mathcal{S}})\varphi\left(H^{(s)}\right) \in \vec{\pi}^s\mathcal{M}_n$$

if and only if $\vec{\pi}^{(s-1)}R^{(s-1)} + \vec{\pi}^{(s-1)}H^{(s)}\gamma(\Pi(\vec{\mathcal{S}})) - \left(\vec{q\pi}\right)^{(s-1)}\Pi(\vec{\mathcal{S}})\varphi\left(H^{(s)}\right) \in \vec{\pi}^s\mathcal{M}_n$ if and only if $H^{(s)}\gamma(\Pi(\vec{\mathcal{S}})) - \vec{q}^{(s-1)}\Pi(\vec{\mathcal{S}})\varphi\left(H^{(s)}\right) \equiv -R^{(s-1)} \bmod \vec{\pi}\mathcal{M}_n$ if and only if $H^{(s)}\Pi^{(0)}(\vec{\mathcal{S}}) - \vec{p}^{(s-1)}\Pi^{(0)}(\vec{\mathcal{S}})\varphi\left(H^{(s)}\right) = -\bar{R}^{(s-1)}$. We notice that $\vec{p}^{i(s-1)}\Pi^{(0)}(\vec{\mathcal{S}})^{-1} \in M_n(\mathcal{O}_E[[\mathcal{S}]]^{|\tau|})$ since $(s-1)i \geq \ell \geq k = \max\{k_0, k_1, ..., k_{f-1}\}$ for any $i = 1, 2, ..., f$.

Let $H = H_1 \times H_2 \times ... \times H_{f-1} \times H_0$ and $-\bar{R}^{(s-1)}\Pi^{(0)}(\vec{\mathcal{S}})^{-1} = R_1 \times R_2 \times ... \times R_{f-1} \times R_0$, the equation $H\Pi^{(0)}(\vec{\mathcal{S}}) - \Pi^{(0)}(\vec{\mathcal{S}})\varphi(H)\,\vec{p}^{(s-1)} = -\bar{R}^{(s-1)}\Pi^{(0)}(\vec{\mathcal{S}})$ is equivalent to the following system of equations in $M_n(\mathcal{O}_E[[\mathcal{S}]]) : H_1 - P_1 H_2(p^{s-1}P_1^{-1}) = R_1$, $H_2 - P_2 H_3(p^{s-1}P_2^{-1}) = R_2, ...\ H_{f-1} - P_{f-1}H_0(p^{s-1}P_{f-1}^{-1}) = R_{f-1}$, $H_0 - P_0 H_1(p^{s-1}P_0^{-1}) = R_0$. We see that $H_1 - QH_1(p^{f(s-1)}Q^{-1}) = R_1 + Q_1R_2(p^{(s-1)}Q_1^{-1}) + Q_2R_3(p^{2(s-1)}Q_2^{-1}) + ... + Q_{f-1}R_0(p^{(s-1)(f-1)}Q_{f-1}^{-1})$, where $Q_i = P_1...P_i$ for all $i = 1, 2, ..., f$ (with $P_f = P_0$ and $Q_f = Q$). The matrices H_i are uniquely determined by H_1 for all $i = 2, ..., f-1, f$, so it suffices to prove that the operator

$$H \longmapsto H - QH(p^{f(s-1)}Q^{-1}) : M_n(\mathcal{O}_E[[\mathcal{S}]]) \longrightarrow M_n(\mathcal{O}_E[[\mathcal{S}]])$$

contains $A = R_1 + Q_1R_2(p^{(s-1)}Q_1^{-1}) + Q_2R_3(p^{2(s-1)}Q_2^{-1}) ... + Q_{f-1}R_0(p^{(s-1)(f-1)}Q_{f-1}^{-1})$ in its image. This is true by assumption (iv) of the lemma. We define $G_\gamma(\vec{\mathcal{S}}) = \lim_{s\to\infty} G_\gamma^{(s)}(\vec{\mathcal{S}})$. ■

Let $\widetilde{M}$ be the ring $M_n(\mathcal{O}_E[[\mathcal{S}]])/I$ where I is the ideal of $M_n(\mathcal{O}_E[[\mathcal{S}]])$ generated by $p \cdot I_n$, and the matrices $X_i \cdot I_n$, where X_i, $i = 0, 1, ..., m-1$ are the indeterminates contained in $\mathcal{S}$. We use the notation of Lemma 3.1 and its proof and we are interested in the image of the operator $\overline{H} \mapsto \overline{H - QH(p^{f\ell}Q^{-1})} : \widetilde{M} \to \widetilde{M}$, where bar denotes reduction mod I.

Proposition 3.2 *If the operator $\overline{H} \mapsto \overline{H - QH(p^{f\ell}Q^{-1})} : \widetilde{M} \to \widetilde{M}$ is surjective, then for each $s \geq \ell + 1$ the operator $H \longmapsto H - QH(p^{f(s-1)}Q^{-1}) : M_n(\mathcal{O}_E[[\mathcal{S}]]) \longrightarrow M_n(\mathcal{O}_E[[\mathcal{S}]])$ is surjective.*

Proof. (i) If $s \geq k+2$. Since $f(s-1) - \sum_{i=0}^{f-1} k_i \geq f(s-1-k) \geq f \geq 1$, since $Q^{-1} = P_0^{-1}P_{f-1}^{-1}P_{f-2}^{-1}...P_1^{-1}$ and $\det(P_i) = C_i p^{k_i}$, we have $p^{f(s-1)}Q^{-1} \in pM_n(\mathcal{O}_E[[\mathcal{S}]])$. Let B be any matrix in $M_n(\mathcal{O}_E[[\mathcal{S}]])$. We can write $B = B - QB\left(p^{f(s-1)}Q^{-1}\right) + pB_1$ for some matrix $B_1 \in M_n(\mathcal{O}_E[[\mathcal{S}]])$. Similarly, $B_1 = B_1 - QB_1\left(p^{f(s-1)}Q^{-1}\right) + pB_2$ for some matrix $B_2 \in M_n(\mathcal{O}_E[[\mathcal{S}]])$. Then $B = (B + pB_1) - Q(B + pB_1)\left(p^{f(s-1)}Q^{-1}\right) + p^2B_2$. Continuing in the same fashion we get

$$B = \left(\sum_{i=0}^{N} p^i B_i\right) - Q\left(\sum_{i=0}^{N} p^i B_i\right)\left(p^{f(s-1)}Q^{-1}\right) + p^{N+1}B_{N+1}$$

for some matrix $B_{N+1} \in M_n(\mathcal{O}_E[[\mathcal{S}]])$, where $B_0 = B$. Let $H = \sum_{i=0}^{\infty} p^i B_i$, then $H \in M_n(\mathcal{O}_E[[\mathcal{S}]])$ and $B = H - QH\left(p^{f(s-1)}Q^{-1}\right)$.

(ii) If ($\ell = k$ and) $s = k+1$. We reduce modulo the ideal I defined before Proposition 3.2. Let A be any element of $M_n(\mathcal{O}_E[[\mathcal{S}]])$. The operator $H \longmapsto H - QH\left(p^{f\ell}Q^{-1}\right) : \widetilde{M} \to \widetilde{M}$ contains $\bar{A} = A \bmod I$ in its image by the assumption of the lemma. Let $A = A_0 - QA_0\left(p^{f\ell}Q^{-1}\right) \bmod I$ for some $A_0 \in M_n(\mathcal{O}_E[[\mathcal{S}]])$. We write $A = A_0 - QA_0\left(p^{f\ell}Q^{-1}\right) + pB_m + X_0B_0 + ... + X_{m-1}B_{m-1}$ for some $B_i \in M_n(\mathcal{O}_E[[\mathcal{S}]])$. Similarly $B_i = B_i^0 - QB_i^0\left(p^{f\ell}Q^{-1}\right) \bmod I$ for some $B_i^0 \in M_n(\mathcal{O}_E[[\mathcal{S}]])$ and for all i. Then

$$\begin{aligned} A &= A_0 - QA_0\left(p^{f\ell}Q^{-1}\right) + pB_m^0 - Q\left(pB_m^0\right)\left(p^{f\ell}Q^{-1}\right) + X_0B_1^0 - Q\left(X_0B_1^0\right)\left(p^{f\ell}Q^{-1}\right) \\ &\quad + ... + X_{m-1}B_{m-1}^0 - Q\left(X_{m-1}B_{f-1}^0\right)\left(p^{f\ell}Q^{-1}\right) \bmod I^2, \text{ therefore} \end{aligned}$$

$A = (A_0 + pB_m^0 + X_0B_1^0 + ... + X_{m-1}B_{m-1}^0) - Q(A + pB_m^0 + X_0B_1^0 + ... + X_{f-1}B_{m-1}^0)\left(p^{f\ell}Q^{-1}\right) \bmod I^2$.

By induction, $A = H - QH\left(p^{f\ell}Q^{-1}\right)$ for some $H \in M_n(\mathcal{O}_E[[\mathcal{S}]])$. ■

The surjectivity assumption of the previous Proposition is usually satisfied due to the following

Proposition 3.3 *If $\ell > k$ or $\ell = k$ and the weights k_i are not all equal, then the operator $\overline{H} \mapsto \overline{H - QH(p^{f\ell}Q^{-1})} : \widetilde{M} \to \widetilde{M}$ is surjective.*

Proof. The statement in the cases where $\ell \geq k+1$ or $\ell = k$ and $k \neq k_i$ for some i follows immediately because $\det Q = Cp^{k_1+k_2+...+k_f}$, where $C = C_1C_2...C_f \in \mathcal{O}_E^{\times}$, since $f\ell > k_1 + ... + k_f$ and $p \in I$. ∎

The following Theorem summarizes the results of the Section. We use the notation of Lemma 3.1.

Theorem 3.4 *Assume that for each $\gamma \in \Gamma_K$ there exists $G_\gamma^{(\ell)} = G_\gamma^{(\ell)}(\vec{X}) \in \mathcal{M}_2$ such that:*
1. $G_\gamma^{(\ell)}(\vec{X}) \equiv I\vec{d} \mod \pi$;
2. $G_\gamma^{(\ell)}(\vec{X})\gamma(\Pi(\vec{X})) - \Pi(\vec{X})\varphi(G_\gamma^{(\ell)}(\vec{X})) \in \vec{\pi}^{\ell}\mathcal{M}_2$;
3. There is no nonzero matrix $H \in M_n(\mathcal{O}_E[[\mathcal{S}]]^{|\tau|})$ such that $HN = p^{ft}NH$ for some $t > 0$;
4. The operator $\overline{H} \mapsto \overline{H - QH(p^{f\ell}Q^{-1})} : \widetilde{M} \to \widetilde{M}$ is surjective.
Then for each $\gamma \in \Gamma_K$, there exists a unique matrix $G_\gamma(\vec{X}) \in \mathcal{M}_2$ such that $G_\gamma(\vec{X}) \equiv I\vec{d} \mod \pi$ and $\Pi(\vec{X})\varphi(G_\gamma(\vec{X})) = G_\gamma(\vec{X})\gamma(\Pi(\vec{X}))$.

For any $\vec{a} = (a_0, a_1, ..., a_{f-1}) \in m_E^f$ we denote $\Pi(\vec{a}) = \Pi_1(a_1) \times \Pi_2(a_2) \times ... \times \Pi_{f-1}(a_{f-1}) \times \Pi_0(a_0)$ the matrix obtained from $\Pi(\vec{X}) = \Pi_1(X_1) \times \Pi_2(X_2) \times ... \times \Pi_{f-1}(X_{f-1}) \times \Pi_0(X_0)$ by substituting $a_i \in m_E$ in the indeterminate X_i of $\Pi_i(X_i)$. We now show how to define Wach modules of rank two over $\mathcal{O}_E[[\pi]]^{|\tau|}$.

Proposition 3.5 *For any $\vec{a} = (a_0, a_1, ..., a_{f-1}) \in m_E^f$ and any $\gamma_1, \gamma_2, \gamma \in \Gamma_K$, the following equations hold: $G_{\gamma_1\gamma_2}(\vec{a}) = G_{\gamma_1}(\vec{a})\gamma_1(G_{\gamma_2}(\vec{a}))$ and $\Pi(\vec{a})\varphi(G_\gamma(\vec{a})) = G_\gamma(\vec{a})\gamma(\Pi(\vec{a}))$.*

Proof. Both the matrices $G_{\gamma_1\gamma_2}(\vec{X})$ and $G_{\gamma_1}(\vec{X})\gamma_1(G_{\gamma_2}(\vec{X}))$ are $\equiv I\vec{d} \mod \pi$ and satisfy $\Pi(\vec{X})\varphi(A) = A\gamma(\Pi(\vec{X}))$. They are equal by the uniqueness part of Lemma 3.1. The second equation follows from conclusion (ii) of the same Lemma. ∎

For any $\vec{a} \in m_E^f$, we equip the module $\boldsymbol{N}(\vec{a}) = \left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta_1 \oplus \left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta_2 \oplus ... \oplus \left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta_n$ with semilinear φ and Γ_K-actions defined by $(\varphi(\eta_1), \varphi(\eta_2), ..., \varphi(\eta_n)) = (\eta_1, \eta_2, ..., \eta_n)\Pi(\vec{a})$ and $(\gamma(\eta_1), \gamma(\eta_2), ..., \gamma(\eta_n)) = (\eta_1, \eta_2, ..., \eta_n)G_\gamma(\vec{a})$, for any $\gamma \in \Gamma_K$. By Proposition 3.5 $(\gamma_1\gamma_2)x = \gamma_1(\gamma_2 x)$, $\varphi(\gamma x) = \gamma(\varphi(x))$ for all $x \in \boldsymbol{N}(\vec{a})$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma_K$ and Γ_K acts trivially on $\boldsymbol{N}(\vec{a})/\pi\boldsymbol{N}(\vec{a})$.

Proposition 3.6 *For any $\vec{a} \in m_E^f$, $\boldsymbol{N}(\vec{a})$ with the φ and Γ_K-actions defined above is a Wach module over $\mathcal{O}_E[[\pi]]^{|\tau|}$ corresponding (by Theorem 1.1) to some G_K-stable $\mathcal{O}_E$-lattice of an n-dimensional crystalline E-representation of G_K with Hodge-Tate weights in $[-k, 0]$.*

Proof. The only thing left to prove is that $q^k\boldsymbol{N}(\vec{a}) \subset \boldsymbol{\varphi}^*(\boldsymbol{N}(\vec{a}))$. Since $\det(\Pi_i) = C_iq^{k_i}$ we have

$\det\Pi(\vec{a}) = (C_1q^{k_1}, C_2q^{k_2}, ..., C_0q^{k_0})$ and $(q^k\eta_1, q^k\eta_2, ..., q^k\eta_n) =$
$(\eta_1, \eta_2, ..., \eta_n)\det\Pi(\vec{a})(C_1^{-1}q^{k-k_1}, C_2q^{k-k_2}, ..., C_0q^{k-k_0}) =$
$(\eta_1, \eta_2, ..., \eta_n)\,(\Pi(\vec{a}) \cdot \mathrm{adj}\,(\Pi(\vec{a})))(C_1^{-1}q^{k-k_1}, C_2q^{k-k_2}, ..., C_0q^{k-k_0}) =$
$(\varphi(\eta_1), \varphi(\eta_2), ..., \varphi(\eta_n)) \cdot (\mathrm{adj}\Pi(\vec{a}))(C_1^{-1}q^{k-k_1}, C_2q^{k-k_2}, ..., C_0q^{k-k_0}) \in \boldsymbol{\varphi}^*(\boldsymbol{N}(\vec{a}))$ ∎

In order to construct Wach modules of n-dimensional crystalline representations of G_K with prescribed labeled Hodge-Tate weights $(W_i)_{\tau_i}$, where $W_i = \{-w_{in-1} \leq ... \leq -w_{i2} \leq -w_{i1} \leq 0\}$ for all $i = 0, 1, ..., f-1$, we consider matrices $\Pi_i \in M_n(\mathcal{O}_E[[\pi, \mathcal{S}]]^{|\tau|})$ with $\det(\Pi_i) = C_iq^{k_i}$, where $C_i \in \mathcal{O}_E^{\times}$ for all i, and apply Theorem 3.4 for the matrix $\Pi(\vec{\mathcal{S}}) = \Pi_1 \times \Pi_2 \times ... \times \Pi_{f-1} \times \Pi_0$. When $K = \mathbb{Q}_p$, in order for the representations corresponding to the family constructed by Π_i to

have labeled Hodge-Tate weights W_i, we must have $k_i = \sum_{j=1}^{n-1} w_{ij}$. This condition is imposed by weak admissibility. If for such matrices Π_i the matrix $\Pi(\vec{\mathcal{S}}) = \Pi_1 \times \Pi_2 \times ... \times \Pi_{f-1} \times \Pi_0$ satisfies the hypothesis of Theorem 3.4, then the corresponding crystalline representations of G_K will have labeled weights $(W_i)_{\tau_i}$. The set of weights W_i is completely determined by the matrix Π_i and is computed applying Theorem 1.1, as illustrated in Proposition 4.15.

We now prove the main theorem concerning modulo p reductions of the crystalline representations corresponding to the families of Wach modules constructed by Theorem 3.6. By reduction modulo p, we mean reduction modulo the maximal ideal m_E of the ring of integers of the coefficient field E. If T is a G_K-stable $\mathcal{O}_E$-lattice in some E-linear representation V of G_K, then we denote $\bar{V} = (k_E \bigotimes_{\mathcal{O}_E} T)^{s.s.}$, where k_E is the residue field of $\mathcal{O}_E$ and $s.s.$ denotes semisimplification. Recall that by a theorem of Brauer and Nesbitt the representation $\bar{V}$ is independent of the lattice T. In the next Theorem the representations $V_{\vec{w},\vec{\alpha}}$ are those constructed in Proposition 3.6. Let $T(\vec{\alpha})$ be a G_K-stable $\mathcal{O}_E$-lattice in $V(\vec{\alpha})$ such that $\boldsymbol{N}(\vec{\alpha}) = N(V(\vec{\alpha})) \cap D(T(\vec{\alpha}))$.

Theorem 3.7 *For any $\vec{\alpha} \in m_E^f$, the isomorphism $\bar{V}(\vec{\alpha}) \simeq \bar{V}(\vec{0})$ holds.*

Proof. We prove that the k_E-linear representations of G_K, $k_E \bigotimes_{\mathcal{O}_E} T(\vec{\alpha})$ and $k_E \bigotimes_{\mathcal{O}_E} T(\vec{0})$ are isomorphic. Let $\vec{a} \in m_E^f$ be such that $\vec{\alpha} = \vec{a} \cdot \vec{z}(\vec{0})$. Since $\Pi(\vec{\mathcal{S}})$ and $G_\gamma(\vec{\mathcal{S}}) \in \mathcal{M}_n$ we have $G_\gamma(\vec{a}) \equiv G_\gamma(\vec{0}) \bmod m_E$ and $\Pi(\vec{a}) \equiv \Pi(\vec{0}) \bmod m_E$. If $D(\vec{a}) = (\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_K) \bigotimes_{\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_K^+} \boldsymbol{N}(\vec{a})$ is the (φ, Γ_K)-module associated to $\boldsymbol{N}(\vec{a})$ (Section 1.1), then $T(\vec{\alpha}) = T(D(\vec{a}))$. As (φ, Γ_K)-modules, $D(\vec{a})/m_E D(\vec{a}) \simeq D(\vec{0})/m_E D(\vec{0})$, hence $T\left(D(\vec{a})/m_E D(\vec{a})\right) \simeq T\left(D(\vec{0})/m_E D(\vec{0})\right)$, where T is Fontaine's functor of Section 1.1. Since Fontaine's functor is exact, $(T(\vec{\alpha})/m_E T(\vec{\alpha})) \simeq \left(T(\vec{0})/m_E T(\vec{0})\right)$. ∎

4 Families of two-dimensional crystalline representations

The main difficulty in applying the theorem above is to construct the matrices $G_\gamma^{(\ell)}(\vec{X})$ which satisfy conditions 1 and 2. When $n = 2$, let E_{ij} the 2×2 matrix with 1 in the (i, j)-entry and 0 everywhere else. If in Proposition 3.3 $\ell = k = k_i$ for all i and $\bar{Q} = Q \bmod I$, then the operator is surjective if $\bar{Q} \in \left\{\bar{0}, -\bar{E}_{12}, -\bar{E}_{21}\right\}$ and is not surjective if $\bar{Q} \in \left\{\bar{E}_{11}, \bar{E}_{22}\right\}$. The proof is a straightforward computation. In the two-dimensional case, we also have the following

Lemma 4.1 *If the matrix $Q_f = P_1 P_2 ... P_{f-1} P_f$ (with $P_f = P_0$ as usual) does not have eigenvalues which are a scalar multiple of each other, then the matrix $N = Nm_\varphi(\Pi^{(0)})$, where $\Pi^{(0)} = P_1 \times P_2 \times ... \times P_{f-1} \times P_0$ satisfies condition (iii) of Lemma 3.1.*

Proof. Let $H \in M_n(\mathcal{O}_E[[\mathcal{S}]]^{|\tau|})$ be a nonzero matrix such $HN = p^{ft} NH$ for some $t > 0$. We write $H = H_1 \times H_2 \times ... \times H_f$ and $N = N_1 \times N_2 \times ... \times N_f$. Since $\Pi^{(0)} \varphi(N) (\Pi^{(0)})^{-1} = N$, $P_i N_{i+1} P_i^{-1} = N_i$ for all i. Since $Q_f = N_1$, none of the N_i has eigenvalues which are a scalar multiple of each other. If H is invertible then $N_1 = Q_f$ has eigenvalues with quotient p^{ft} which contradicts the assumption of the Lemma. If H is not invertible and nonzero, then there exists index i such that $H_i N_i = p^{ft}$

N_iH_i and rank$(H_i) = 1$. There exists invertible matrix B such that $BH_iB^{-1} = \begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix}$ with $(\alpha, \beta) \neq (0,0)$. Let $\Gamma = BN_iB^{-1}$ and write $\Gamma = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$. Then $H_iN_i = p^{ft}\ N_iH_i$ is equivalent to $p^{ft}\Gamma BH_iB^{-1} = BH_iB^{-1}\Gamma$ which implies that $c_2 = 0$ and $p^{ft}c_1\alpha = \alpha c_1$. If $\alpha \neq 0$, then $c_1 = 0$ contradiction since Γ is invertible. If $\alpha = 0$, then $p^{ft}\beta c_4 = \beta c_1$ and $p^{ft}c_4 = c_1$ (since $\beta \neq 0$). Then Γ has two eigenvalues with quotient p^{ft}. Then the matrix N_i and its conjugate $Q_f = N_1$ have eigenvalues with quotient p^{ft} which contradicts the assumption of the Lemma. Hence $H = 0$. ■

Thanks to this, one can instead of checking condition (iii) of Theorem 3.4, check that the matrix Q does not have eigenvalues which are a scalar multiple of each other, which is often more convenient. In practice the matrices $G_\gamma^{(\ell)}$ will be diagonal. In this case, if we replace $\Pi(\vec{X})$ by its conjugate $\Pi^B(\vec{X}) = B^{-1}\Pi(\vec{X})B$, where $B = B_1 \times ... \times B_{f-1} \times B_0$ with $B_i =$diag$(1, u_i)$ with $u_i \in \mathcal{O}_E^\times$, and if $G_\gamma^{(\ell)}(\vec{X})\gamma(\Pi(\vec{X})) - \Pi(\vec{X})\varphi(G_\gamma^{(\ell)}(\vec{X})) \in \vec{\pi}^\ell\mathcal{M}_2$, then $G_\gamma^{(\ell)}(\vec{X})\gamma(\Pi^B(\vec{X})) - \Pi^B(\vec{X})\varphi(G_\gamma^{(\ell)}(\vec{X})) \in \vec{\pi}^\ell\mathcal{M}_2$. We will be applying the method outlined in Theorem 3.4 with the matrices Π_i being conjugates by matrices B_i as above, of any of the following eight basic types of matrices.

Definition 4.2

$$\mathbf{T}_1\text{: } \begin{pmatrix} 0 & -1 \\ C_iq^{k_i} & X_iz_i \end{pmatrix} \quad \mathbf{T}_2\text{: } \begin{pmatrix} X_i\varphi(z_i) & -1 \\ C_iq^{k_i} & 0 \end{pmatrix} \quad \mathbf{T}_3\text{: } \begin{pmatrix} X_iz_i & C_iq^{k_i} \\ -1 & 0 \end{pmatrix} \quad \mathbf{T}_4\text{: } \begin{pmatrix} 0 & C_iq^{k_i} \\ -1 & X_i\varphi(z_i) \end{pmatrix}$$

$$\mathbf{T}_5\text{: } \begin{pmatrix} C_iq^{k_i} & 0 \\ X_i\varphi(z_i) & 1 \end{pmatrix} \quad \mathbf{T}_6\text{: } \begin{pmatrix} C_iq^{k_i} & X_iz_i \\ 0 & 1 \end{pmatrix} \quad \mathbf{T}_7\text{: } \begin{pmatrix} 1 & X_i\varphi(z_i) \\ 0 & C_i^{k_i}q^{k_i} \end{pmatrix} \quad \mathbf{T}_8\text{: } \begin{pmatrix} 1 & 0 \\ X_iz_i & C_iq^{k_i} \end{pmatrix}$$

where X_i is an indeterminate, $C_i \in \mathcal{O}_E^\times$ and z_i is the degree $\leq \ell - 1$ polynomial part of some formal power series in $\mathbb{Z}_p[[\pi]]$ such that $z_i \equiv p^{m_\ell} \bmod \pi$, where $m_\ell = \lfloor \frac{\ell-1}{p-1} \rfloor$. The polynomial z_i is allowed to be zero.

The list above is certainly not complete. One, for example, can replace any Π_i by any conjugate by some matrix of $GL_2(\mathcal{O}_E[[\pi, X_i]])$. To keep the construction of the matrices $G_\gamma^{(\ell)}$ reasonably simple we restrict ourselves to the eight cases defined above and only consider their conjugates by matrices of the form diag$(1, u_i)$ with $u_i \in \mathcal{O}_E^\times$. Let $\Pi(\vec{X})$ be as in Theorem 3.4. We write $\Pi^{\vec{i}}(\vec{X}) = \Pi_1(X_1) \times \Pi_2(X_2) \times ... \times \Pi_{f-1}(X_{f-1}) \times \Pi_0(X_0)$ with $\vec{i} = (i_1, i_2, ..., i_{f-1}, i_0)$ the vector in $\{1, 2, ..., 8\}^f$ whose j-th coordinate i_j is the type of the matrix Π_j for all $j \in I_0$.

4.1 Some families or rank-two Wach modules

In this Section, we apply the method outlined in Theorem 3.4 and consider matrices $\Pi_i(X_i)$ of types T_1,T_2,T_3,T_4. Examples involving more types of matrices are studied in Section 6. Let $\Pi(\vec{X}) = \Pi^{\vec{i}}(\vec{X})$ be as in the previous section and $\ell \geq k = \max\{k_0, k_1, ..., k_{f-1}\}$. For each $\gamma \in \Gamma_K$ we will define functions x_i^γ, $y_i^\gamma \in 1 + \pi\mathbb{Z}_p[[\pi]]$ so that, if $G_\gamma^{(\ell)} = \begin{pmatrix} \vec{x}^\gamma & \vec{0} \\ \vec{0} & \vec{y}^\gamma \end{pmatrix}$ with $\vec{x}^\gamma = (x_0^\gamma, x_1^\gamma, ..., x_{f-1}^\gamma)$ and $\vec{y}^\gamma = (y_0^\gamma, y_1^\gamma, ..., y_{f-1}^\gamma)$, then $G_\gamma^{(\ell)}\gamma(\Pi(\vec{X})) - \Pi(\vec{X})\varphi(G_\gamma^{(\ell)}) \in \vec{\pi}^\ell\mathcal{M}_2$. Let

$$\Pi(\vec{X})\varphi(G_\gamma^{(\ell)}) - G_\gamma^{(\ell)}\gamma(\Pi(\vec{X})) = \begin{pmatrix} \vec{A} & \vec{B} \\ \vec{\Gamma} & \vec{\Delta} \end{pmatrix}.$$

We demand that the coordinates of $\vec{A},\vec{B},\vec{\Gamma},\vec{\Delta}$ which contain none of the indeterminates X_i be zero and the rest of them belong to $\pi^{\ell}\mathcal{O}_E[[\pi,X_0,...,X_{f-1}]]$. Let

$$\Pi(\vec{X})=\begin{pmatrix} (\alpha_1X_1,\alpha_2X_2...\alpha_{f-1}X_{f-1},\alpha_0X_0) & (\beta_1\ ,\ \ \beta_2\ ,\ \ .\ \ .\ \ .\ \ ,\ \beta_{f-1}\ ,\beta_0\ ) \\ (\gamma_1,\ \ \ \gamma_2\ ,\ \ .\ \ .\ \ .\ \ ,\ \gamma_{f-1},\ \ \ \gamma_0\ ) & (\delta_1X_1,\delta_2X_2...\delta_{f-1}X_{f-1},\delta_0X_0) \end{pmatrix}$$

with $\{\beta_i,\gamma_i\}=\{-1,C_iq^{k_i}\}$ and $\{\alpha_i,\delta_i\}=\{0,z_i,\varphi(z_i)\}$. If for each $\gamma\in\Gamma_K$ the functions $x_i^{\gamma},y_i^{\gamma}$ are defined to be the unique $\equiv 1 \bmod \pi$ solution of the following system of equations:
$\beta_1\varphi(y_1^{\gamma})=x_0^{\gamma}(\gamma\beta_1),\ \gamma_1\varphi(x_1^{\gamma})=y_0^{\gamma}(\gamma\gamma_1),\ \beta_2\varphi(y_2^{\gamma})=x_1^{\gamma}(\gamma\beta_2),\ \gamma_2\varphi(x_2^{\gamma})=y_1^{\gamma}(\gamma\gamma_2),$
....
$\beta_{f-1}\varphi(y_{f-1}^{\gamma})=x_{f-2}^{\gamma}(\gamma\beta_{f-1}),\ \gamma_{f-1}\varphi(x_{f-1}^{\gamma})=y_{f-2}^{\gamma}(\gamma\gamma_{f-1}),\ \beta_0\varphi(y_0^{\gamma})=x_{f-1}^{\gamma}(\gamma\beta_0),$
$\gamma_0\varphi(x_0^{\gamma})=y_{f-1}^{\gamma}(\gamma\gamma_0)$.
Then $\Pi(\vec{X})\varphi(G_{\gamma}^{(\ell)})-G_{\gamma}^{(\ell)}\gamma(\Pi(\vec{X}))=\begin{pmatrix}\vec{A} & \vec{0}\\ \vec{0} & \vec{\Delta}\end{pmatrix}$, where $\vec{A}=((\alpha_1\varphi(x_1^{\gamma})-x_0^{\gamma}(\gamma\alpha_1))X_1,...,(\alpha_0\varphi(x_0^{\gamma})-x_{f-1}^{\gamma}(\gamma\alpha_0))X_0)$ and $\vec{\Delta}=((\delta_1\varphi(y_1^{\gamma})-y_0^{\gamma}(\gamma\delta_1))X_1,...,(\delta_0\ \varphi(y_0^{\gamma})-y_{f-1}^{\gamma}(\gamma\delta_0))X_0)$. For $i\in I_0$ we define

$$r_{\beta_i}=\begin{cases} k_i & \text{if } \beta_i=C_iq^{k_i},\\ 0 & \text{if } \beta_i=-1\end{cases}\quad \text{and } r_{\gamma_i}=\begin{cases} k_i & \text{if } \gamma_i=C_iq^{k_i},\\ 0 & \text{if } \gamma_i=-1\end{cases}$$

We use the convention that $\beta_i=\beta_j$ and $\gamma_i=\gamma_j$ whenever $i\equiv j \bmod f$. The solution of the system above is obtained using Lemma 2.3 and a not-so-short, but straightforward computation which we skip and is the following:

Case 1. If f is even,
$x_0^{\gamma}=(\lambda_{f,\gamma})^{r_{\beta_1}}\left(\varphi(\lambda_{f,\gamma})\right)^{r_{\gamma_2}}\left(\varphi^2(\lambda_{f,\gamma})\right)^{r_{\beta_3}}...\left(\varphi^{f-2}(\lambda_{f,\gamma})\right)^{r_{\beta_{f-1}}}\left(\varphi^{f-1}(\lambda_{f,\gamma})\right)^{r_{\gamma_0}}$

$y_0^{\gamma}=(\lambda_{f,\gamma})^{r_{\gamma_1}}\left(\varphi(\lambda_{f,\gamma})\right)^{r_{\beta_2}}\left(\varphi^2(\lambda_{f,\gamma})\right)^{r_{\gamma_3}}...\left(\varphi^{f-2}(\lambda_{f,\gamma})\right)^{r_{\gamma_{f-1}}}\left(\varphi^{f-1}(\lambda_{f,\gamma})\right)^{r_{\beta_0}}$

$x_i^{\gamma}=(\frac{\beta_{i+1}}{\gamma\beta_{i+1}})\varphi(\frac{\gamma_{i+2}}{\gamma\gamma_{i+2}})\varphi^2(\frac{\beta_{i+3}}{\gamma\beta_{i+3}})...\varphi^{f-i-2}(\frac{\gamma_{f-1}}{\gamma\gamma_{f-1}})\varphi^{f-i-1}(\frac{\beta_0}{\gamma\beta_0})\varphi^{f-i}(y_0^{\gamma})$ for $i=1,3,...,f-1$,

$x_i^{\gamma}=(\frac{\beta_{i+1}}{\gamma\beta_{i+1}})\varphi(\frac{\gamma_{i+2}}{\gamma\gamma_{i+2}})\varphi^2(\frac{\beta_{i+3}}{\gamma\beta_{i+3}})...\varphi^{f-i-2}(\frac{\beta_{f-1}}{\gamma\beta_{f-1}})\varphi^{f-i-1}(\frac{\gamma_0}{\gamma\gamma_0})\varphi^{f-i}(x_0^{\gamma})$ for $i=2,4,...,f-2$,

$y_i^{\gamma}=(\frac{\gamma_{i+1}}{\gamma\gamma_{i+1}})\varphi(\frac{\beta_{i+2}}{\gamma\beta_{i+2}})\varphi^2(\frac{\gamma_{i+3}}{\gamma\gamma_{i+3}})...\varphi^{f-i-2}(\frac{\beta_{f-1}}{\gamma\beta_{f-1}})\varphi^{f-i-1}(\frac{\gamma_0}{\gamma\gamma_0})\varphi^{f-i}(x_0^{\gamma})$ for $i=1,3,...,f-1$,

$y_i^{\gamma}=(\frac{\gamma_{i+1}}{\gamma\gamma_{i+1}})\varphi(\frac{\beta_{i+2}}{\gamma\beta_{i+2}})\varphi^2(\frac{\gamma_{i+3}}{\gamma\gamma_{i+3}})...\varphi^{f-i-2}(\frac{\gamma_{f-1}}{\gamma\gamma_{f-1}})\varphi^{f-i-1}(\frac{\beta_0}{\gamma\beta_0})\varphi^{f-i}(y_0^{\gamma})$ for $i=2,4,...,f-2$.

Case 2. If f is odd,
$x_0^{\gamma}=(\lambda_{2f,\gamma})^{r_{\beta_1}}\left(\varphi(\lambda_{2f,\gamma})\right)^{r_{\gamma_2}}\left(\varphi^2(\lambda_{2f,\gamma})\right)^{r_{\beta_3}}\times...\times\left(\varphi^{f-3}(\lambda_{2f,\gamma})\right)^{r_{\beta_{f-2}}}\left(\varphi^{f-2}(\lambda_{2f,\gamma})\right)^{r_{\gamma_{f-1}}}$
$\times(\varphi^{f-1}(\lambda_{2f,\gamma}))^{r_{\beta_0}}(\varphi^{f}(\lambda_{2f,\gamma}))^{r_{\gamma_1}}(\varphi^{f+1}(\lambda_{2f,\gamma}))^{r_{\beta_2}}(\varphi^{f+2}(\lambda_{2f,\gamma}))^{r_{\gamma_3}}$
$\times...\times\varphi^{2f-2}(\lambda_{2f,\gamma}))^{r_{\beta_{f-1}}}(\varphi^{2f-1}(\lambda_{2f,\gamma}))^{r_{\gamma_0}}$

$x_i^{\gamma}=(\frac{\beta_{i+1}}{\gamma\beta_{i+1}})\varphi(\frac{\gamma_{i+2}}{\gamma\gamma_{i+2}})\varphi^2(\frac{\beta_{i+3}}{\gamma\beta_{i+3}})...\varphi^{f-i-2}(\frac{\beta_{f-1}}{\gamma\beta_{f-1}})\varphi^{f-i-1}(\frac{\gamma_0}{\gamma\gamma_0})\varphi^{f-i}(x_0^{\gamma})$ for $i=1,3,...f-2$,

$x_i^{\gamma}=(\frac{\beta_{i+1}}{\gamma\beta_{i+1}})\varphi(\frac{\gamma_{i+2}}{\gamma\gamma_{i+2}})\varphi^2(\frac{\beta_{i+3}}{\gamma\beta_{i+3}})...\varphi^{2f-i-2}(\frac{\beta_{f-1}}{\gamma\beta_{f-1}})\varphi^{2f-i-1}(\frac{\gamma_0}{\gamma\gamma_0})\varphi^{2f-i}(x_0^{\gamma})$ for $i=2,4,...,f-1$,

$y_i^{\gamma}=(\frac{\gamma_{i+1}}{\gamma\gamma_{i+1}})\varphi(\frac{\beta_{i+2}}{\gamma\beta_{i+2}})\varphi^2(\frac{\gamma_{i+3}}{\gamma\gamma_{i+3}})...\varphi^{f-i-2}(\frac{\beta_{f-1}}{\gamma\beta_{f-1}})\varphi^{f-i-1}(\frac{\gamma_0}{\gamma\gamma_0})\varphi^{f-i}(x_0^{\gamma})$ for $i=0,2,...,f-1$,

$y_i^{\gamma}=(\frac{\gamma_{i+1}}{\gamma\gamma_{i+1}})\varphi(\frac{\beta_{i+2}}{\gamma\beta_{i+2}})\varphi^2(\frac{\gamma_{i+3}}{\gamma\gamma_{i+3}})...\varphi^{2f-i-2}(\frac{\beta_{f-1}}{\gamma\beta_{f-1}})\varphi^{2f-i-1}(\frac{\gamma_0}{\gamma\gamma_0})\varphi^{2f-i}(x_0^{\gamma})$ for $i=1,3,...,f-2$.

Lemma 4.3 *Let $k = \max\{k_0, k_1, ..., k_{f-1}\}$, $\ell \geq k$ and $m_\ell = \lfloor \frac{\ell-1}{p-1} \rfloor$. For each $\gamma \in \Gamma_K$ and $i \in I_0$, the functions x_i^γ, y_i^γ, λ_f and $\lambda_{f,\gamma} \in \mathbb{Q}_p[[\pi]]$ have the following properties:*
(1) $\lambda_f(0) = 1$;
(2) $\lambda_{f,\gamma}$, $x_i^\gamma, y_i^\gamma \in 1 + \pi\mathbb{Z}_p[[\pi]]$;

(3a) *There exist $z_i \in \mathbb{Z}_p[\pi]$ with $\deg_\pi z_i \leq \ell - 1$ such that $z_i \equiv p^{m_\ell} \bmod \pi$ and $z_i\varphi(y_i^\gamma) - y_{i-1}^\gamma\gamma(z_i) \in \pi^\ell\mathbb{Z}_p[[\pi]]$ for all $\gamma \in \Gamma_K$;*

(3b) *If $\beta_i = -1, \gamma_i = C_i q^{k_i}$, there exist $z_i \in \mathbb{Z}_p[\pi]$ with $\deg_\pi z_i \leq \ell - 1$ such that $z_i \equiv p^{m_\ell} \bmod \pi$ and $\varphi(z_i)\varphi(x_i^\gamma) - x_{i-1}^\gamma\varphi(\gamma z_i) \in \pi^\ell\mathbb{Z}_p[[\pi]]$ for all $\gamma \in \Gamma_K$;*

(3c) *There exist $z_i \in \mathbb{Z}_p[\pi]$ with $\deg_\pi z_i \leq \ell - 1$ such that $z_i \equiv p^{m_\ell} \bmod \pi$ and $z_i\varphi(x_i^\gamma) - x_{i-1}^\gamma(\gamma z_i) \in \pi^\ell\mathbb{Z}_p[[\pi]]$ for all $\gamma \in \Gamma_K$;*

(3d) *If $\beta_i = C_i q^{k_i}, \gamma_i = -1$, there exist $z_i \in \mathbb{Z}_p[\pi]$ with $\deg_\pi z_i \leq \ell - 1$ such that $z_i \equiv p^{m_\ell} \bmod \pi$ and $\varphi(z_i)\varphi(y_i^\gamma) - y_{i-1}^\gamma(\gamma z_i) \in \pi^\ell\mathbb{Z}_p[[\pi]]$ for all $\gamma \in \Gamma_K$.*

Proof. (1) and $(2)_a$ have been proven in 2.2. From the same Lemma it is clear that $x_i^\gamma, y_i^\gamma \in 1 + \pi\mathbb{Z}_p[[\pi]]$ for all γ and i. (3a) We notice that the coefficients of π in $\frac{q}{p}$ have the property $v_p(a_i) + \frac{i}{p-1} \geq 0$ for all $i = 0, 1, ...$ Motivated by this we consider the set of all functions of $\mathbb{Q}_p[[\pi]]$ with the same property:

$$\mathcal{R} = \{\sum_{i\geq 0} a_i\pi^i \in \mathbb{Q}_p[[\pi]] : v_p(a_i) + \frac{i}{p-1} \geq 0 \text{ for all } i \geq 0\}.$$

This is a ring with the obvious operations, stable under φ and Γ_K. One easily checks that $\frac{p}{q} \in \mathcal{R}$ and therefore $\frac{q_n}{p}$ and $\frac{p}{q_n} \in \mathcal{R}$ for all $n \geq 1$. We now prove the existence of z_i as in case (3a). Since $y_i^\gamma \in 1 + \pi\mathbb{Z}_p[[\pi]]$ for all $\gamma \in \Gamma_K$, it suffices to prove the existence of a z_i with the desired properties, such that $z_i - \frac{y_{i-1}^\gamma}{\varphi(y_i^\gamma)}\gamma(z_i) \in \pi^\ell\mathbb{Z}_p[[\pi]]$ for all $\gamma \in \Gamma_K$. (1) Assume that f is even and i is odd. Then $y_i^\gamma = (\frac{\gamma_{i+1}}{\gamma\gamma_{i+1}})\varphi(\frac{\beta_{i+2}}{\gamma\beta_{i+2}})\varphi^2(\frac{\gamma_{i+3}}{\gamma\gamma_{i+3}})...\varphi^{f-i-1}(\frac{\gamma_0}{\gamma\gamma_0})\varphi^{f-i}(x_0^\gamma)$, $y_{i-1}^\gamma = (\frac{\gamma_i}{\gamma\gamma_i})\varphi(\frac{\beta_{i+1}}{\gamma\beta_{i+1}})\varphi^2(\frac{\gamma_{i+2}}{\gamma\gamma_{i+2}})...\varphi^{f-i}(\frac{\beta_0}{\gamma\beta_0})\varphi^{f-i+1}(y_0^\gamma)$ and $\frac{y_{i-1}^\gamma}{\varphi(y_i^\gamma)} = B\gamma(B^{-1})$, where

$$B = \frac{\gamma_i\varphi(\beta_{i+1})...\varphi^{f-i}(\beta_0)\varphi^{f-i+1}\left((\lambda_f)^{r_{\gamma_1}}(\varphi(\lambda_f))^{r_{\beta_2}}...(\varphi^{f-2}(\lambda_f))^{r_{\gamma_{f-1}}}(\varphi^{f-1}(\lambda_f))^{r_{\beta_0}}\right)}{\varphi(\gamma_{i+1})...\varphi^{f-i}(\gamma_0)\varphi^{f-i+1}\left((\lambda_f)^{r_{\beta_1}}(\varphi(\lambda_f))^{r_{\gamma_2}}...(\varphi^{f-2}(\lambda_f))^{r_{\beta_{f-1}}}(\varphi^{f-1}(\lambda_f))^{r_{\gamma_0}}\right)}$$

Since the $(\frac{q}{p})^{\pm 1}, \lambda_f^{\pm 1}$ are contained in $\mathcal{R}$ and are $\equiv 1 \bmod \pi$, and since $\{\beta_j, \gamma_j\} = \{-1, C_j q^{k_j}\}$ for all j, after replacing $C_j q^{k_j}$ by $C_j(\frac{q}{p})^{k_j}p^{k_j}$ we can write $B = Cp^N B^*$ for some $C \in \mathcal{O}_E^\times$, $N \in \mathbb{Z}$ and $B^* \in \mathcal{R}$ with $B^* \equiv 1 \bmod \pi$. Since Γ_K acts trivially on $\mathcal{O}_E^\times$ we may replace B by $C^{-1}B$ and assume that $C = 1$. We write $B = p^N B^*$. Let $p^M B = z_i + A$, where $A \in \pi^\ell\mathbb{Q}_p[[\pi]]$ and $\deg_\pi z_i \leq \ell - 1$, for integer M which will be chosen large enough so that $z_i \in \mathbb{Z}_p[\pi]$. Since $p^{M+N}B^* = z_i + A$ and $B^* \in \mathcal{R}$, $v_p(z_i^j) - M - N + \frac{j}{p-1} \geq 0$ for all $j \geq 0$. We need $v_p(z_i^j) > -1$ for all $j = 0, 1, ..., \ell - 1$ so it suffices to have $M + N - \frac{\ell-1}{p-1} > -1$. We choose $M = \lfloor \frac{\ell-1}{p-1} \rfloor - N$. Then $z_i \in \mathbb{Z}_p[\pi]$, $\deg_\pi z_i \leq \ell - 1$ and $z_i \equiv p^{M+N} = p^{m_\ell} \bmod \pi$, where $m_\ell = \lfloor \frac{\ell-1}{p-1} \rfloor$. The definition of z_i is independent of $\gamma \in \Gamma_K$ for all i. For each $\gamma \in \Gamma_K$, $z_i - \frac{y_{i-1}^\gamma}{\varphi(y_i^\gamma)}\gamma(z_i) = z_i - B\gamma(B^{-1})\gamma(z_i) = p^M B - A - B\gamma(B^{-1})[p^M(\gamma B) - \gamma A] =$

$\gamma A - A \in \pi^{\ell}\mathbb{Q}_p[[\pi]]$. Since $z_i \in \mathbb{Z}_p[\pi]$ and $B\gamma(B^{-1}) = \frac{y_{i-1}^{\gamma}}{\varphi(y_i^{\gamma})} \in 1 + \pi\mathbb{Z}_p[[\pi]]$, $z_i\varphi(y_i^{\gamma}) - y_{i-1}^{\gamma}\gamma(z_i) \in \pi^{\ell}\mathbb{Z}_p[[\pi]] = \mathbb{Z}_p[[\pi]] \cap \pi^{\ell}\mathbb{Q}_p[[\pi]]$ for all $\gamma \in \Gamma_K$. The cases with f even and i even, with f odd and i odd, or with f odd and i even are proven similarly. $(3b)$ Since $\beta_i = -1$, $x_{i-1}^{\gamma} = \varphi(y_i^{\gamma})$ for all $\gamma \in \Gamma_K$ and it suffices to prove the existence or a z_i with the desired properties satisfying $z_i x_i^{\gamma} - x_{i-1}^{\gamma}(\gamma z_i) \in \pi^{\ell}\mathbb{Z}_p[[\pi]]$ for all $\gamma \in \Gamma_K$. This is proven arguing as in case $(3a)$. The case $(3c)$ is proven arguing as in case $(3a)$. For the case $(3d)$, we notice that $y_{i-1}^{\gamma} = \varphi(x_i^{\gamma})$ for all γ and proceed as in case $(3b)$. $\blacksquare$

The following technical corollary, whose proof is contained in the previous Lemma, will be used in Sections 6 and 7.

Corollary 4.4 *(1)* $\mathcal{R} = \{\sum_{i\geq 0} a_i\pi^i \in \mathbb{Q}_p[[\pi]] : v_p(a_i) + \frac{i}{p-1} \geq 0$ *for all* $i \geq 0\}$ *is a subring of* $\mathbb{Q}_p[[\pi]]$ *stable under the* φ *and the* Γ_K*-actions;*
(2) $(\frac{q_n}{p})^{\pm 1} \in \mathcal{R}$ *for all* $n \geq 1$ *and* $(\lambda_f)^{\pm 1} \in \mathcal{R}$ *for all* $f \geq 1$;
(3) Let $B = Cp^N B^*$, *where* $C \in \mathcal{O}_E^{\times}$, $N \in \mathbb{Z}$ *and* $B^* \in \mathcal{R} \setminus \{0\}$ *be such that* $\frac{B}{\gamma B} \in 1 + \pi\mathbb{Z}_p[[\pi]]$ *for any* $\gamma \in \Gamma_K$. *Let* $k \geq 1$ *be any fixed integer. There exists* $z = z(k, B) \in \mathbb{Z}_p[\pi]$ *with* $\deg_{\pi} z \leq k-1$ *and* $z \equiv p^{\lfloor \frac{k-1}{p-1} \rfloor} \bmod \pi$ *such that* $z - \gamma z \frac{B}{\gamma B} \in \pi^k \mathbb{Z}_p[[\pi]]$ *for any* $\gamma \in \Gamma_K$.

Definition 4.5 *For* $i \in I_0$ *we define matrices* $\Pi_i = \Pi_{i,k_i}^{C_i, z_i}(X_i) \in M_2(\mathcal{O}_E[[\pi, X_i]])$, *where the* X_i *are indeterminates as follows:*

$\Pi_i = \begin{pmatrix} 0 & -1 \\ C_i q^{k_i} & X_i z_i \end{pmatrix}$ *for any* z_i *having properties as in case* $(3a)$ *of Lemma 4.3 and any* $C_i \in \mathcal{O}_E^{\times}$,

$\Pi_i = \begin{pmatrix} X_i\varphi(z_i) & -1 \\ C_i q^{k_i} & 0 \end{pmatrix}$ *for any* z_i *having properties as in case* $(3b)$ *of Lemma 4.3 and any* $C_i \in \mathcal{O}_E^{\times}$,

$\Pi_i = \begin{pmatrix} X_i z_i & C_i q^{k_i} \\ -1 & 0 \end{pmatrix}$ *for any* z_i *having properties as in case* $(3c)$ *of Lemma 4.3 and any* $C_i \in \mathcal{O}_E^{\times}$,

$\Pi_i = \begin{pmatrix} 0 & C_i q^{k_i} \\ -1 & X_i\varphi(z_i) \end{pmatrix}$ *for any* z_i *having properties as in case* $(3d)$ *of Lemma 4.3 and any* $C_i \in \mathcal{O}_E^{\times}$.

We define $\Pi(\vec{X}) = \Pi_1 \times \Pi_2 \times ... \times \Pi_{f-1} \times \Pi_0 \in \mathcal{M}_2 = M_2(\mathcal{O}_E[[\pi, X_0, ..., X_{f-1}]]^{|\tau|})$. $\square$

Corollary 4.6 *For each* $\gamma \in \Gamma_K$ *there exists a matrix* $G_{\gamma}^{(\ell)} \in \mathcal{M}_2$ *such that (i)* $G_{\gamma}^{(\ell)} \equiv I\vec{d} \bmod \pi$ *and (ii)* $G_{\gamma}^{(\ell)}\gamma(\Pi(\vec{X})) - \Pi(\vec{X})\varphi(G_{\gamma}^{(\ell)}) \in \vec{\pi}^{\ell}\mathcal{M}_2$.

Definition 4.7 *We define sets*
$C_1 = \{(T_1, T_2), (T_1, T_3), (T_4, T_2), (T_4, T_3)\}$, $C_2 = \{(T_2, T_1), (T_3, T_1), (T_2, T_4), (T_3, T_4)\}$,
$E_1 = \{(T_1, T_3), (T_1, T_4), (T_2, T_3), (T_2, T_4)\}$, $E_2 = \{(T_3, T_1), (T_4, T_1), (T_3, T_2), (T_4, T_2)\}$,
$\mathcal{C}_1 = \{(1,2), (1,3), (4,2), (4,3)\}$, $\mathcal{C}_2 = \{(2,1), (3,1), (2,4), (3,4)\}$,
$\mathcal{E}_1 = \{(1,3), (1,4), (2,3), (2,4)\}$, $\mathcal{E}_2 = \{(3,1), (4,1), (3,2), (4,2)\}$.
We write $(P_1, P_2) = (T_i, T_j)$ *if and only if* P_1 *is of type* T_i *and* P_2 *of type* T_j. *We retain this notation throughout.*

Proposition 4.8 *The operator* $\overline{H} \mapsto \overline{H - QH(p^{f\ell}Q^{-1})} : \widetilde{M} \to \widetilde{M}$ *is surjective unless* f *is even,* $k = k_i$ *for all* $i, \ell = k$, *and either (i)* $(P_i, P_{i+1}) \in E_1$ *for all* $i = 1, 3, ..., f-1$, *in which case the image consists of elements of* $\widetilde{M}$ *with* $(1,2)$ *entry* 0, *or (ii)* $(P_i, P_{i+1}) \in E_2$ *for all* $i = 1, 3, ..., f-1$, *in which case the image consists of elements of* $\widetilde{M}$ *with* $(2,1)$ *entry* 0.

Proof. Follows from Proposition 3.3 and induction. ■

Proposition 4.9 *Q_f has eigenvalues which are a scalar multiple of each other if and only if f is even and either $(P_i, P_{i+1}) \in C_1$ for all $i = 1, 3, ..., f-1$ or $(P_i, P_{i+1}) \in C_2$ for all $i = 1, 3, ..., f-1$.*

This follows from the following series of lemmas:

Lemma 4.10 *If P_1, P_2 are matrices of type $T_i, i = 1, 2, 3, 4$ and $Q_2 = P_1P_2 = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$, then Q_2 has one of the following forms: (A) $Q_2 = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$ with $\alpha, \delta \in \mathcal{O}_E \setminus \{0\}$ and $\gamma = \lambda_1 X_1 + \lambda_2 X_2$ with $\lambda_i \in \mathcal{O}_E \setminus \{0\}$ if and only if $(P_1, P_2) \in C_1$. (B) $Q_2 = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$ with $\alpha, \delta \in \mathcal{O}_E \setminus \{0\}$ and $\beta = \lambda_1 X_1 + \lambda_2 X_2$ with $\lambda_i \in \mathcal{O}_E \setminus \{0\}$ if and only if $(P_1, P_2) \in C_2$. (C) If $(P_1, P_2) \notin C_1 \cup C_2$, then either (i) $\alpha = \lambda X_1 X_2 + \mu$, $\lambda, \mu \in \mathcal{O}_E \setminus \{0\}$, $\delta \in \mathcal{O}_E \setminus \{0\}, \beta \in O_E[X_1]$ has degree 1 and $\gamma \in \mathcal{O}_E[X_2]$ has degree 1 or (ii) $\delta = \lambda X_1 X_2 + \mu$, $\lambda, \mu \in \mathcal{O}_E \setminus \{0\}$, $\alpha \in \mathcal{O}_E \setminus \{0\}, \gamma \in \mathcal{O}_E[X_1]$ has degree 1 and $\beta \in \mathcal{O}_E[X_2]$ has degree 1.*

Proof. We just compute. ■

Lemma 4.11 *Let $Q_i = P_1P_2...P_i$ for $i = 1, 2, ..., f$ (with $P_f = P_0$). If $i \geq 2$ is even and $Q_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_\iota \end{pmatrix}$, then $\alpha_i \delta_i \neq 0$.*

Proof. If $i = 2$, follows from the previous Lemma. Assume $i > 2$ and the Lemma holds for $i - 2$. Let $P_{i-1}P_i = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then $Q_i = \begin{pmatrix} \alpha\alpha_{i-2} + \gamma\beta_{i-2} & * \\ * & \beta\gamma_{i-2} + \delta\delta_{i-2} \end{pmatrix}$. If $P_{i-1}P_i$ is as in case (A) of the previous Lemma, then the $(1, 1)$ entry of the matrix Q_i has positive degree in X_{i-1} and X_i if $\beta_{i-2} \neq 0$, and equals $\delta\delta_{i-2}$ which is nonzero by the inductive hypothesis, if $\beta_{i-2} = 0$. Similarly the $(2, 2)$ entry of Q_i is nonzero. The remaining cases when $P_{i-1}P_i$ is as in cases (B), $(C)(i)$ or $(C)(ii)$ of the previous Lemma are proven similarly. ■

Lemma 4.12 *If f is odd, Q_f does not have eigenvalues which are a scalar multiple of each other.*

Proof. Since the determinant of Q_f is a scalar, the eigenvalues of Q_f are a scalar multiple of each other if and only if $Tr(Q_f)$ is a scalar. Let $Q_f = Q_{f-1}P_f$, where $Q_{f-1} = \begin{pmatrix} \alpha_{f-1} & \beta_{f-1} \\ \gamma_{f-1} & \delta_{f-1} \end{pmatrix}$ with $\alpha_{f-1}\delta_{f-1} \neq 0$ (see previous Lemma). If P_f is of type 1, then $Tr(Q_f) = p^{k_i} C_i \beta_{f-1} + p^{m_i} X_i \delta_{f-1} - \gamma_{f-1}$ which is non-scalar since $\delta_{f-1} \neq 0$. The cases when P_f is of type $2, 3$ or 4 are proven similarly. ■

Lemma 4.13 *If $f \geq 2$ is even and Q_f has eigenvalues which are a scalar multiple of each other, then (i) If $P_f \in \{T_1, T_4\}$, then $P_{f-1} \in \{T_2, T_3\}$ and Q_{f-2} has eigenvalues which are a scalar multiple of each other. (ii) If $P_f \in \{T_2, T_3\}$, then $P_{f-1} \in \{T_1, T_4\}$ and Q_{f-2} has eigenvalues which are a scalar multiple of each other.*

Proof. As in the proof of the previous Lemma, Q_f has eigenvalues which are a scalar multiple of each other if and only if $Tr(Q_f)$ is a scalar. Let $Q_{f-1} = \begin{pmatrix} \alpha_{f-1} & \beta_{f-1} \\ \gamma_{f-1} & \delta_{f-1} \end{pmatrix}$. If P_f is

of type 1, then $Q_f = \begin{pmatrix} p^{k_i}C_i\beta_{f-1} & p^{m_i}X_i\beta_{f-1}-\alpha_{f-1} \\ p^{k_i}C_i\delta_{f-1} & p^{m_i}X_i\delta_{f-1}-\gamma_{f-1} \end{pmatrix}$ which implies that $\delta_{f-1}=0$. Hence $\begin{pmatrix} \alpha_{f-2} & \beta_{f-2} \\ \gamma_{f-2} & \delta_{f-2} \end{pmatrix} P_{f-1} = \begin{pmatrix} \alpha_{f-1} & \beta_{f-1} \\ \gamma_{f-1} & 0 \end{pmatrix}$ from which we see that P_{f-1} has to be of type 2 or 3, given that $\alpha_{f-2}\delta_{f-2} \neq 0$ by Lemma 4.11. Since $Q_f = \begin{pmatrix} p^{k_i}C_i\beta_{f-1} & p^{m_i}X_i\beta_{f-1}-\alpha_{f-1} \\ 0 & -\gamma_{f-1} \end{pmatrix}$ and $P_{f-1}P_f = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$ with α, δ scalars, $Tr(Q_{f-2})$ is a scalar. Since $\det(Q_{f-2})$ is a scalar, Q_{f-2} has eigenvalues which are a scalar multiple of each other. The proof when P_f is of type 2 is similar and the proof of (ii) is identical to the proof of (i). ■

Proof. (Of Proposition 4.9)**.** One direction follows from the previous Lemma. The other direction follows from Lemma 4.10. ■

4.2 The corresponding families of crystalline representations

Let $\Pi^{\vec{i}}(\vec{X}) = \Pi_1(X_1) \times \Pi_2(X_2) \times \ldots \times \Pi_{f-1}(X_{f-1}) \times \Pi_0(X_0)$ with $\vec{i} \in \{1,2,3,4\}^f$ and $\Pi_i(X_i)$ as in Definition 4.5. The definition of the matrices Π_i and $P_i = \Pi_i \bmod \pi$ depends on the choice of the z_i in Lemma 4.3 and therefore on ℓ. (I) If $[K:\mathbb{Q}_p]$ is even, we exclude those vectors $\vec{\imath}$ with $(i_j, i_{j+1}) \in \mathcal{C}_1$ for all $j \in \{1,3,5,\ldots,f-1\}$ or $(i_j, i_{j+1}) \in \mathcal{C}_2$ for all $j \in \{1,3,5,\ldots,f-1\}$. These cases will be studied separately in Proposition 5.15. If the integers k_i are all equal and either $(i_j, i_{j+1}) \in \mathcal{E}_1$ for all $j \in \{1,3,5,\ldots,f-1\}$ or $(i_j, i_{j+1}) \in \mathcal{E}_2$ for all $j \in \{1,3,5,\ldots,f-1\}$, we define $\ell = k+1$. In any other case we define $\ell = k$. (II) If $[K:\mathbb{Q}_p]$ is odd, $\vec{i}$ can be any vector in $\{1,2,3,4\}^f$. We define $\ell = k$ and we make no further assumptions.

Proposition 4.14 *For any $\gamma \in \Gamma_K$, there exists unique matrix $G_\gamma(\vec{X}) = G_\gamma^{\vec{i}}(\vec{X}) \in \mathcal{M}_2$ such that (i) $G_\gamma(\vec{X}) \equiv I\vec{d} \bmod \pi$ and (ii) $\Pi^{\vec{i}}(\vec{X})\varphi(G_\gamma(\vec{X})) = G_\gamma(\vec{X})\gamma(\Pi^{\vec{i}}(\vec{X}))$.*

Proof. Conditions 1 and 2 of Theorem 3.4 are satisfied by Corollary 4.6. By the choice of the vectors $\vec{i}$, condition 3 of of Theorem 3.4 is satisfied. By the choice of the vectors $\vec{i}$ and the integer ℓ condition 4 of Theorem 3.4 is satisfied and the proposition follows. ■

For any $\vec{a} \in m_E^f$, we equip the module $\boldsymbol{N}^{\vec{i}}(\vec{a}) = \left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta_1 \oplus \left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta_2$ with semilinear φ and Γ_K-actions defined as in Proposition 3.6.

Let $\Pi^{\vec{i}}(\vec{X})$ be as in the previous section. For any $\vec{a} \in m_E^f$ we consider the matrices of $GL_2\left(E^{|\tau|}\right)$ obtained from the matrices $P^{\vec{i}}(\vec{a}) = P_1(X_1) \times P_2(X_2) \times \ldots \times P_{f-1}(X_{f-1}) \times P_0(X_0)$ by substituting $X_j = a_j$ in $P_j(X_j)$. We define families of filtered φ-modules $D^{\vec{i}}_{\vec{w},\vec{a}} = \left(E^{|\tau|}\right)\eta_1 \oplus \left(E^{|\tau|}\right)\eta_2$ with Frobenius endomorphisms defined by $(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2)P^{\vec{i}}(\vec{a})$ and filtrations defined by

$$\mathrm{Fil}^j(D^{\vec{i}}_{\vec{w},\vec{a}}) = \begin{cases} E^{|\tau|}\eta_1 \oplus E^{|\tau|}\eta_2 & \text{if } j \leq 0, \\ E^{|\tau_{I_0}|}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 \leq j \leq w_0, \\ E^{|\tau_{I_1}|}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 + w_0 \leq j \leq w_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ E^{|\tau_{I_{t-1}}|}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 + w_{t-2} \leq j \leq w_{t-1}, \\ 0 & \text{if } j \geq 1 + w_{t-1}, \end{cases}$$

where $\vec{x}=(x_0,x_1,...,x_{f-1})$ and $\vec{y}=(y_0,y_1,...,y_{f-1})$, with

$$(x_i,y_i)=\begin{cases}(1,0) & \text{if } P_i \text{ is } \mathrm{T}_1,\\ (1,\alpha_i) & \text{if } P_i \text{ is } \mathrm{T}_2,\\ (0,1) & \text{if } P_i \text{ is } \mathrm{T}_3,\\ (\alpha_i,1) & \text{if } P_i \text{ is } \mathrm{T}_4,\\ (1,-\alpha_i) & \text{if } P_i \text{ is } \mathrm{T}_5,\\ (1,0) & \text{if } P_i \text{ is } \mathrm{T}_6,\\ (-\alpha_i,0) & \text{if } P_i \text{ is } \mathrm{T}_7,\\ (0,1) & \text{if } P_i \text{ is } \mathrm{T}_8,\end{cases} \tag{F}$$

and $\alpha_i = a_i z_i(0)$. If instead of Π_i we use its conjugate $B_i^{-1}\Pi_i B_i$, where B_i =diag$(1,u_i)$ with $u_i\in\mathcal{O}_E^{\times}$, we replace (x_i,y_i) by $(x_i,u_i^{-1}y_i)$, or equivalently by (u_ix_i,y_i).

Proposition 4.15 *If the matrix $\Pi^{\vec{i}}(\vec{X})$ satisfies the conclusions of Theorem 3.4, then for any $\vec{a}\in m_E^f$ the filtered φ-modules $(D^{\vec{i}}_{\vec{w},\vec{a}},\varphi)$ defined above are weakly admissible and*

$$D^{\vec{i}}_{\vec{w},\vec{a}}\simeq E^{|\tau|}\bigotimes_{\mathcal{O}_E^{|\tau|}}(\boldsymbol{N}^{\vec{i}}_{\vec{w}}(\vec{a})/\pi\boldsymbol{N}^{\vec{i}}_{\vec{w}}(\vec{a}))$$

as filtered φ-modules, where $\boldsymbol{N}^{\vec{i}}_{\vec{w}}(\vec{a})$ is as in Proposition 3.6.

Proof. By Theorem 1.1, $\vec{x}\eta_1+\vec{y}\eta_2\in\mathrm{Fil}^j(\boldsymbol{N}^{\vec{i}}_{\vec{w}}(\vec{a}))$ if and only if $\varphi(\vec{x})\ \varphi(\eta_1)+\ \varphi(\vec{y})\ \varphi(\eta_2)\in q^j\boldsymbol{N}^{\vec{i}}_{\vec{w}}(\vec{a})$ or equivalently

$$e_i\varphi(\vec{x})\varphi(\eta_1)+e_i\varphi(\vec{y})\varphi(\eta_2)\in q^je_i\boldsymbol{N}^{\vec{i}}_{\vec{w}}(\vec{a})\ for\ all\ i\in I_0\ (\bigstar).$$

We fix some $i\in I_0$ and do the calculation in the case where Π_i is of type 4. Then $\Pi_i(a_i)=\begin{pmatrix}0 & C_iq^{k_i}\\ -1 & a_i\varphi(z_i)\end{pmatrix}$ and $(\bigstar)$ is equivalent to $\begin{cases}q^j\mid-\varphi(y_i)q^{k_i},\\ q^j\mid\varphi(-x_i+y_ia_iz_i).\end{cases}$ We use that $q^j\mid\varphi(x)$ if and only if $\pi^j\mid x$ for any $x\in\mathcal{O}_E[[\pi]]$. If $j\geq 1+k_i$, then $x_i,y_i\equiv 0 \bmod \pi$. Let $1\leq j\leq k_i$, then the system above is equivalent to $\pi^j\mid -x_i+y_ia_iz_i$. Since $a_iz_i\equiv\alpha_i \bmod \pi$,

$$e_i\vec{x}\eta_1+e_i\vec{y}\eta_2+\pi\boldsymbol{N}^{\vec{i}}_{\vec{w}}(\vec{a})=\begin{cases}\alpha_i\bar{y}_ie_i\eta_1+\bar{y}_ie_i\eta_2+\pi\boldsymbol{N}^{\vec{i}}_{\vec{w}}(\vec{a}) & \text{if } 1\leq\ j\leq k_i,\\ \\ 0\ \text{ if }\ j\geq k_i\end{cases}$$

where $\bar{y}_i=y_i \bmod \pi$ can be any element of $\mathcal{O}_E$. Since

$$\mathrm{Fil}^0(\boldsymbol{N}^{\vec{i}}_{\vec{w}}(\vec{a})/\pi\boldsymbol{N}^{\vec{i}}_{\vec{w}}(\vec{a}))=(\mathcal{O}_E^{|\tau|})\eta_1\bigoplus(\mathcal{O}_E^{|\tau|})\eta_2,$$

we get

$$e_i\mathrm{Fil}^j(\boldsymbol{N}^{\vec{i}}_{\vec{k}}(\vec{a})/\pi\boldsymbol{N}^{\vec{i}}_{\vec{k}}(\vec{a}))=\begin{cases}e_i(\mathcal{O}_E^{|\tau|})\eta_1\bigoplus e_i(\mathcal{O}_E^{|\tau|})\eta_2 \text{ if } j\leq 0,\\ e_i(\mathcal{O}_E^{|\tau|})(\vec{x}^i\eta_1+\vec{y}^i\eta_2) \text{ if } 1\leq j\leq k_i,\\ 0 \text{ if } j\geq 1+k_i,\end{cases}$$

with $e_i\vec{x}^i = (0, ..., x_i, ..., 0)$, $e_i\vec{y}^i = (0, ..., y_i, ..., 0)$ and $(x_i, y_i) = (\alpha_i, 1)$. Calculating for the other choices of $\Pi_i(a_i)$, we see that for all $i \in I_0$, (x_i, y_i) is as in (F). Arguing as in the proof of Proposition 2.4, we get

$$\mathrm{Fil}^j(\boldsymbol{N}^{\vec{i}}_{\vec{w}}(\vec{a})/\pi\boldsymbol{N}^{\vec{i}}_{\vec{w}}(\vec{a})) = \begin{cases} (\mathcal{O}_E^{|\tau|})\eta_1 \oplus (\mathcal{O}_E^{|\tau|})\eta_2 & \text{if } j \leq 0, \\ (\mathcal{O}_E^{|\tau|})f_{I_0}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 \leq j \leq w_0, \\ (\mathcal{O}_E^{|\tau|})f_{I_1}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 + w_0 \leq j \leq w_1, \\ \cdots\cdots\cdots & \\ (\mathcal{O}_E^{|\tau|})f_{I_{t-1}}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 + w_{t-2} \leq j \leq w_{t-1}, \\ \quad 0 \ \text{ if } j \geq 1 + w_{t-1} & \end{cases}$$

with $\vec{x} = (x_0, x_1, ..., x_{f-1})$ and $\vec{y} = (y_0, y_1, ..., y_{f-1})$ and (x_i, y_i) as in (F). The isomorphism

$$D^{\vec{i}}_{\vec{k},\vec{a}} \simeq E^{|\tau|} \bigotimes_{\mathcal{O}_E^{|\tau|}} (\boldsymbol{N}^{\vec{i}}_{\vec{k}}(\vec{a})/\pi\boldsymbol{N}^{\vec{i}}_{\vec{k}}(\vec{a}))$$

is now obvious. Proving that if replace one of the Π_i by $B_i^{-1}\Pi_i B_i$, where $B_i = \mathrm{diag}(1, u_i)$ with $u_i \in \mathcal{O}_E^{\times}$, then we replace (x_i, y_i) by $(x_i, u_i^{-1}y_i)$ is a simple computation. ■

The crystalline representation corresponding to corresponding to $D^{\vec{i}}_{\vec{w},\vec{a}}$ is denoted $V^{\vec{i}}_{\vec{w},\vec{\alpha}}$, where $\vec{\alpha} = \vec{a} \cdot \vec{z}(\vec{0})$, with $\vec{z} = (z_1, ..., z_{f-1}, z_0)$.

4.3 Construction of the split-reducible two-dimensional crystalline representations

In the previous sections we gave examples where the Galois action on the Wach module was implicitly constructed by some inductive argument. In this section we construct Wach modules with labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$ whose Γ_K-action can be explicitly written down. The corresponding crystalline representations turn out to be split-reducible. Let k_i be non negative integers, $C_i \in \mathcal{O}_E^{\times}$ for $i = 1, 2$ and $\vec{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_{f-1})$, $\vec{\beta} = (\beta_0, \beta_1, ..., \beta_{f-1})$ be vectors with coordinates such that $\{\alpha_i, \beta_i\} = \{0, k_i\}$ for all $i \in I_0$. Let

$$\Pi = \Pi^{C_1,\vec{\alpha}}_{C_2,\vec{\beta}} = \begin{pmatrix} (C_1q^{\alpha_1}, q^{\alpha_2}, ..., q^{\alpha_{f-1}}, q^{\alpha_0}) & (\ 0\ ,\ 0, \ . \ . \ . \ , 0\ ,\ 0\ ) \\ (\ 0\ ,\ 0, \ . \ . \ . \ , 0\ ,\ 0\ ) & (C_2q^{\beta_1}, q^{\beta_2}, ..., q^{\beta_{f-1}}, q^{\beta_0}) \end{pmatrix}$$

For each $\gamma \in \Gamma_K$, let $G_\gamma = \begin{pmatrix} (x_0^\gamma, x_1^\gamma, ..., x_{f-1}^\gamma) & (0,\ 0,\ . \ . \ . \ ,\ 0) \\ (0,\ 0,\ . \ . \ . \ ,\ 0) & (y_0^\gamma, y_1^\gamma, ..., y_{f-1}^\gamma) \end{pmatrix}$, where $x_i^\gamma, y_i^\gamma \in \mathcal{O}_E[[\pi]]$. We'll define $x_i^\gamma, y_i^\gamma \in \mathcal{O}_E[[\pi]]$ with the following properties : (i) $x_i^\gamma, y_i^\gamma \equiv 1 \bmod \pi$ for all i and γ, (ii) $\Pi\varphi(G_\gamma) = G_\gamma\gamma(\Pi)$ for all $\gamma \in \Gamma_K$. By Lemma 2.3 there is a unique solutions in x_i^γ, y_i^γ congruent to $1 \bmod \pi$ given by

$$x_i^\gamma = (\frac{q}{\gamma q})^{A_i}(\varphi^{f-i}(\lambda_{f,\gamma}))^{\alpha_1}(\varphi^{f-i+1}(\lambda_{f,\gamma}))^{\alpha_2}...(\varphi^{2f-i-2}(\lambda_{f,\gamma}))^{\alpha_{f-1}}(\varphi^{2f-i-1}(\lambda_{f,\gamma}))^{\alpha_0},$$
$$y_i^\gamma = (\frac{q}{\gamma q})^{B_i}(\varphi^{f-i}(\lambda_{f,\gamma}))^{\beta_1}(\varphi^{f-i+1}(\lambda_{f,\gamma}))^{\beta_2}...(\varphi^{2f-i-2}(\lambda_{f,\gamma}))^{\beta_{f-1}}(\varphi^{2f-i-1}(\lambda_{f,\gamma}))^{\beta_0},$$

where $A_i = \alpha_{i+1} + \alpha_{i+2} + ... + \alpha_f$ and $B_i = \beta_{i+1} + \beta_{i+2} + ... + \beta_f$ for all $i \in I_0$, with the conventions that $\alpha_i = \alpha_j$ and $\beta_i = \beta_j$ whenever $i \equiv j \bmod f$. We have proven that, for the matrix Π defined as

above and for each $\gamma \in \Gamma_K$, there exists a unique diagonal matrix $G_\gamma \in M_2\left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)$ such that (i) $G_\gamma \equiv \vec{Id} \mod \pi$ and (ii) $\Pi\varphi(G_\gamma) = G_\gamma\gamma(\Pi)$. We have the following

Proposition 4.16 *Let* $\boldsymbol{N} = \left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta_1 \bigoplus \left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta_2$ *with semilinear* φ *and* Γ_K*-actions defined by* Π *and* G_γ *respectively. The module* $\boldsymbol{N}$ *is a two-dimensional Wach module over* $\mathcal{O}_E[[\pi]]^{|\tau|}$ *with labeled Hodge-Tate weights* $\{0, -k_i\}_{\tau_i}$. *Let* (D, φ) *be the filtered* φ*-module defined by the matrix*

$$P_\varphi = P^{C_1,\vec{\alpha}}_{C_2,\vec{\beta}}(\varphi) = \begin{pmatrix} (C_1p^{\alpha_1}, p^{\alpha_2}, ..., p^{\alpha_{f-1}}, p^{\alpha_0}) & (\;0\;,\;0,\;.\;.\;.\;,0\;,\;0\;) \\ (\;0\;,\;0,\;.\;.\;.\;,0\;,\;0\;) & (C_2p^{\beta_1}, p^{\beta_2}, ..., p^{\beta_{f-1}}, p^{\beta_0}) \end{pmatrix}$$

with filtrations as in Proposition 5.1 with $\vec{x} = (x_0, x_1, ..., x_{f-1})$ *and* $\vec{y} = (y_0, y_1, ..., y_{f-1})$, *where*

$$(x_i, y_i) = \begin{cases} (0,1) \text{ if } \beta_i = k_i > 0, \\ (1,0) \text{ otherwise.} \end{cases}$$

Then

$$(D, \varphi) \simeq E^{|\tau|} \bigotimes_{\mathcal{O}_E^{|\tau|}} (\boldsymbol{N}/\pi\boldsymbol{N})$$

as filtered φ*-modules. The corresponding crystalline representations are split-reducible.*

Proof. The proof is similar to that of Proposition 2.4. Split-reducibility follows from Proposition 5.1. ∎

Proposition 4.17 *All the split-reducible, two-dimensional* E*-representations of* G_K *with labeled Hodge-Tate weights* $\{0, -k_i\}_{\tau_i}$ *are those constructed in Proposition 4.16.*

Proof. This follows immediately from Proposition 5.1, as a split-reducible representation is necessarily F-semisimple. ∎

5 Reductions of two-dimensional crystalline representations

This section is devoted to explicit computations of reductions of two-dimensional crystalline representations. In parts, we follow [3] closely. Before we treat the two-dimensional case we take care of the reductions of crystalline characters.

5.1 The crystalline characters modulo p

Let $C \in \mathcal{O}_E^\times$, and $\chi_{C,\vec{w}}$ be the crystalline character corresponding to the Wach module $\boldsymbol{N}_{C,\vec{w}} = \left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta$ with φ action defined by $\varphi(e) = (Cq^{k_1}, q^{k_2}, ..., q^{k_{f-1}}, q^{k_0})\eta$ and the unique commuting with it Γ_K-action defined in Proposition 2.4. When $C = 1$ we simply write $\chi_{\vec{w}}$. The crystalline character χ_{e_i} has labeled Hodge-Tate weights $-e_{i+1}$ for all i (see Proposition 2.4). By taking tensor products we see that

$$\chi_{C,\vec{w}} = \chi_{C,\vec{0}} \cdot \chi_{e_0}^{k_1} \cdot \chi_{e_1}^{k_2} \cdot ... \cdot \chi_{e_{f-2}}^{k_{f-1}} \cdot \chi_{e_{f-1}}^{k_0}.$$

One easily sees that $\chi_{C,\vec{0}}$ is the unramified character of G_K which maps Frob_K (geometric Frobenius) to $\sqrt[f]{C}$. We compose the embeddings of K into E with the embedding $E \hookrightarrow \bar{\mathbb{Q}}_p$ that we fixed in the introduction and get the embeddings, which we still denote τ_i of K in $\bar{\mathbb{Q}}_p$. We denote

$\bar{\tau}_i : k_K \to \bar{\mathbb{F}}_p$ the mod p reduction of such an embedding τ_i, where k_K is the residue field of K. Let $\omega_{f,\bar{\tau}_i}$ be the fundamental character of I_K defined by composing the embedding $\bar{\tau}_i$ with the homomorphism $I_K \to k_K^{\times}$ gotten from local class field theory, with uniformizers corresponding to geometric Frobenius elements. By Fontaine-Laffaille, theory it is known that $(\bar{\chi}_{e_i})_{|I_K} = \omega_{f,\bar{\tau}_{i+1}}^{-1}$ for $i = 0, 1, ..., f-1$; τ_i as in Section 1 (c.f. [7], Lemma 3.8).
In this section, we give a description of $(\bar{\chi}_{e_i})_{|I_K}$ using the Wach module of χ_{e_i}. First, we compute the $\bar{\mathbb{F}}_p^{\times}$-valued characters $\bar{\chi}_{\vec{w}}$ of G_K subject to the condition that $1 + p + p^2 + ... + p^{f-1} \mid k_1 + pk_2 + ... + p^{f-1}k_0$. We search for elements $\vec{\alpha} \cdot \bar{\eta} \in k_E((\pi)))^{|\tau|} \bigotimes\limits_{k_E[[\pi]]^{|\tau|}} \bar{N}_{C,\vec{w}}$ such that $\varphi(\vec{\alpha} \cdot \bar{\eta}) = (\lambda, \lambda, ..., \lambda)\vec{\alpha} \cdot \bar{\eta}$, for some $\lambda \in k_E^{\times}$. If $\vec{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_{f-1})$, we easily see that this is equivalent to $\lambda^f \alpha_0 = \bar{C}\pi^{(p-1)(k_1+pk_2+...+p^{f-1}k_0)}\varphi^f(\alpha_0)$, $\varphi(\alpha_0)\pi^{k_0(p-1)} = \lambda\alpha_{f-1}$, $\varphi(\alpha_{f-1})\pi^{k_{f-1}(p-1)} = \lambda\alpha_{f-2}, ...,$ $\varphi(\alpha_2)\pi^{k_2(p-1)} = \lambda\alpha_1$, $\varphi(\alpha_1)\pi^{k_1(p-1)} = \lambda\alpha_0$. Let $\lambda^f = \bar{C}$. Since $1 + p + p^2 + ... + p^{f-1} \mid k_1 + pk_2 + ... + p^{f-1}k_0$ the equation $\lambda^f \alpha_0 = \bar{C}\pi^{(p-1)(k_1+pk_2+...+p^{f-1}k_0)}\varphi^f(\alpha_0)$ has unique (up to a nonzero constant) nonzero solution given by $\alpha_0 = \pi^{-\frac{(k_1+pk_2+...+p^{f-1}k_0)}{1+p+...+p^{f-1}}}$. The vector $\vec{\alpha}$ is such that $\varphi(\vec{\alpha} \cdot \bar{\eta}) = (\lambda, \lambda, ..., \lambda)\vec{\alpha} \cdot \bar{\eta}$ and if $\varphi(\vec{\alpha}' \cdot \bar{\eta}) = (\lambda, \lambda, ..., \lambda)\vec{\alpha}' \cdot \bar{\eta}$, for some nonzero vector $\vec{\alpha}'$, then the formulas above and the uniqueness (up to a constant) of the solution α_0 imply that $\vec{\alpha}' = (t, t, ..., t)\vec{\alpha}$, for some $t \in k_E^{\times}$. Let $\gamma \in G_K$ and $\gamma(\vec{\alpha} \cdot \bar{\eta}) = \vec{\beta}_{\gamma} \cdot \bar{\eta}$. Commutativity of φ and γ implies that $\varphi(\vec{\beta}_{\gamma} \cdot \bar{\eta}) = (\lambda, \lambda, ..., \lambda)\vec{\beta}_{\gamma} \cdot \bar{\eta}$. If $\vec{\beta}_{\gamma} = (\beta_0, \beta_1, ..., \beta_{f-1})$, then $\beta_i = t\alpha_i$ for some $t \in k_E^{\times}$ and all i. Since $\beta_0 = \gamma\alpha_0$ and $\alpha_0 = \pi^{-\frac{(k_1+pk_2+...+p^{f-1}k_0)}{1+p+...+p^{f-1}}}$, $t = \omega(\gamma)^{-\frac{(k_1+pk_2+...+p^{f-1}k_0)}{1+p+...+p^{f-1}}}$, where ω is the mod p cyclotomic character of G_K. We have $\varphi(\vec{\alpha} \cdot \bar{\eta}) = (\lambda, \lambda, ..., \lambda)\vec{\alpha} \cdot \bar{\eta}$ and $\gamma(\vec{\alpha} \cdot \bar{\eta}) = \omega(\gamma)^{-\frac{(k_1+pk_2+...+p^{f-1}k_0)}{1+p+...+p^{f-1}}}(\vec{\alpha} \cdot \bar{\eta})$ for all $\gamma \in G_K$. Therefore, if $1 + p + p^2 + ... + p^{f-1} \mid k_1 + pk_2 + ... + p^{f-1}k_0$, then

$$\bar{\chi}_{C,\vec{w}} = \mu_{\lambda} \cdot \omega^{-\frac{(k_1+pk_2+...+p^{f-1}k_0)}{1+p+...+p^{f-1}}},$$

where μ_{λ} is the unramified character of G_K which sends Frob_K to $\lambda = \sqrt[f]{\bar{C}}$. It follows immediately that $\bar{\chi}_{\vec{1}} = \omega^{-1}$. For each $i \in I_0$, $\bar{\chi}_{\vec{1}} = \bar{\chi}_{(1+p+p^2+...+p^i,0,0,...,0,1,1,...,1)}$, (where the first 1 appears in the $i+1$-th position), hence $\bar{\chi}_{\vec{1}} = \bar{\chi}_{e_1}^{1+p+...+p^i} \cdot \bar{\chi}_{(0,0,...,0,1,...,1)}$ and $\bar{\chi}_{e_1}^{1+p+...+p^i} = \bar{\chi}_{(1,1,...,1,0,...,0)}$ for any $i \in I_0$. For $i = 1$ we get $\bar{\chi}_{e_1}^{1+p} = \bar{\chi}_{(1,1,0,...,0)}$ which implies that $\bar{\chi}_{e_1}^{p} = \bar{\chi}_{e_2}$. Similarly $\bar{\chi}_{e_1}^{p^i} = \bar{\chi}_{e_{i+1}}$ for any $i \in I_0$. It is clear that $\bar{\chi}_{e_1}^{\frac{p^f-1}{p-1}} = \omega^{-1}$. When $p \neq 2$, since Γ_K is procyclic, $\bar{\chi}_{e_1}(\gamma) = \psi(\gamma)\omega_f^{-1}(\gamma)$ for any $\gamma \in I_K$, where $\omega_f(\gamma) = \frac{\gamma(\sqrt[p^f-1]{-p})}{\sqrt[p^f-1]{-p}}$ for any $\gamma \in I_K$, for some character $\psi : I_K \to \mathbb{F}_{p^f}^{\times}$ with $\psi^{\frac{p^f-1}{p-1}} = \mathbf{1}$.

5.2 Classification of two-dimensional crystalline representations with coefficients

In this section recall a classification of F-semisimple, two-dimensional, crystalline E-linear representations of G_K with labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$, by classifying up to isomorphism the corresponding weakly admissible filtered φ-modules with labeled Hodge-Tate weights as above. The results of the section will be subsequently used at various places. For their proofs see [12]. When $K = \mathbb{Q}_p$, the characteristic polynomial is enough to determine the isomorphism class of a two dimensional crystalline representation of G_K. When $K \neq \mathbb{Q}_p$ it completely fails to do so. Even

worse, there can exist infinitely many non isomorphic irreducible two-dimensional crystalline representations of G_K sharing the same characteristic polynomial and filtration (Corollary 5.3). The situation is detailed in this Section. Wach modules corresponding to such families are constructed in Section 6, where several reductions modulo p of the corresponding crystalline representations are explicitly computed.

5.2.1 Rank two weakly admissible filtered φ-modules.

In the following Proposition, we list the rank two F-semisimple, non-scalar filtered φ-modules over $E^{|\tau|}$ with labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$, give a criterion for weak admissibility, and determine the types of the corresponding crystalline representations.

Proposition 5.1 *Let (D, φ) be a rank two F-semisimple, non-scalar filtered φ-module over $E^{|\tau|}$ with labeled Hodge-Tate weights $\{0, -k_i\}_{\tau_i}$. After enlarging E if necessary, there exists an ordered base $\underline{\eta} = (\eta_1, \eta_2)$ of D over $E^{|\tau|}$ with respect to which the matrix of Frobenius takes the form $[\varphi]_{\underline{\eta}} = diag(\vec{\alpha}, \vec{\delta})$ for some vectors $\vec{\alpha}, \vec{\delta} \in (E^{\times})^{|\tau|}$ with $Nm_{\varphi}(\vec{\alpha}) \neq Nm_{\varphi}(\vec{\delta})$.*
The filtration (for the same base) has the form

$$Fil^j(D) = \begin{cases} E^{|\tau|}\eta_1 \oplus E^{|\tau|}\eta_2 & \text{if } j \leq 0, \\ E^{|\tau_{I_0}|}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 \leq j \leq w_0, \\ E^{|\tau_{I_1}|}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 + w_0 \leq j \leq w_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ E^{|\tau_{I_{t-1}}|}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 + w_{t-2} \leq j \leq w_{t-1}, \\ \quad 0 \quad \text{if } j \geq 1 + w_{t-1} & \end{cases}$$

for some vectors $\vec{x}, \vec{y} \in E^{|\tau|}$ with $(x_i, y_i) \neq (0, 0)$ for all $i \in I_0$. We refer to such a base $\underline{\eta}$ as a standard base of (D, φ). The Frobenius-fixed submodules are 0, D, $D_1 = E^{|\tau|}\eta_1$ and $D_2 = E^{|\tau|}\eta_2$. The module D is weakly admissible if and only if

$$(*)\ v_p(Nm_{\varphi}(\vec{\alpha})Nm_{\varphi}(\vec{\delta})) = \sum_{i \in I_0} k_i;\ (*)\ v_p(Nm_{\varphi}(\vec{\alpha})) \geq \sum_{\{i \in I_0:\ y_i = 0\}} k_i$$
$$\text{and } (*)\ v_p(Nm_{\varphi}(\vec{\delta})) \geq \sum_{\{i \in I_0:\ x_i = 0\}} k_i.$$

Assuming that D is weakly admissible,
($\star$) The filtered φ-module D is irreducible if and only if both the inequalities above are strict;
($\star$) The filtered φ-module D is split-reducible if and only if both the inequalities are equalities, or equivalently $I_0^+ \cap J_{\vec{x}} \cap J_{\vec{y}} = \varnothing$. In this case, the only nontrivial weakly admissible submodules are $D_i = E^{|\tau|}\eta_i$, $i = 1, 2$ and $D = D_1 \oplus D_2$;
($\star$) In any other case the filtered φ-module D is reducible, non-split. $\square$

If in the Proposition above $v_p(Nm_{\varphi}(\vec{\alpha})) = \sum_{\{i \in I_0:\ y_i = 0\}} k_i$, then the only nontrivial weakly admissible submodule is (D_1, φ). If $v_p(Nm_{\varphi}(\vec{\delta})) = \sum_{\{i \in I_0:\ x_i = 0\}} k_i$, then the only nontrivial weakly admissible submodule is (D_2, φ). An F-semisimple, weakly admissible filtered φ-module (D, φ) is scalar if and only if all the k_i are zero and $v_p(Nm_{\varphi}(\vec{\alpha})) = 0$. The corresponding crystalline representation is the sum of two unramified characters. This case is trivial and will usually be discarded. If (D, φ) is not

F-semisimple, after extending E if necessary, there exists ordered base $\underline{\eta} = (\eta_1, \eta_2)$ of D over $E^{|\tau|}$ with respect to which the matrix of Frobenius takes the form $[\varphi]_{\underline{\eta}} = \begin{pmatrix} \vec{\alpha} & \vec{0} \\ \vec{\gamma} & \vec{\alpha} \end{pmatrix}$ for some vectors $\vec{\alpha} \in (E^{\times})^{|\tau|}$ and $\vec{\gamma} \in E^{|\tau|}$ with $\vec{\Gamma}_{\vec{\gamma}} = \vec{1}$, where $\vec{\Gamma}_{\vec{\gamma}}$ is the $(2,1)$ entry of the matrix $Nm_{\varphi}([\varphi]_{\underline{\eta}})$. The filtration (for the same base) has the shape of Proposition 5.1. The filtered φ-module (D, φ) is weakly admissible if any only if $2v_p(Nm_{\varphi}(\vec{\alpha})) = \sum\limits_{i \in I_0} k_i$ and $v_p(Nm_{\varphi}(\vec{\delta})) \geq \sum\limits_{\{i \in I_0 : x_i = 0\}} k_i$. The corresponding crystalline representation is irreducible if and only if the inequality is strict and reducible, non-split otherwise. In this case, the only φ-stable weakly admissible submodule is (D_2, φ). Unless otherwise stated, we will only be considering F-semisimple, non-scalar filtered φ-modules.

5.2.2 Isomorphism classes

If (D, φ) is a rank two F-semisimple filtered φ-module over $E^{|\tau|}$, we may without loss of generality assume that $\vec{y} = f_{J\vec{y}}$ i.e. that the nonzero coordinates of $\vec{y}$ equal 1. We may further consider the base $\underline{\zeta} = (\zeta_1, \zeta_2)$ defined by $\zeta_1 = (\sum\limits_{i \in J'_{\vec{x}}} e_i + \sum\limits_{i \in J_{\vec{x}}} x_i^{-1} e_i)\eta_1$ and $\zeta_2 = \eta_2$ and assume that $\vec{x} = f_{J_{\vec{x}}}$ and $\vec{y} = f_{J_{\vec{y}}}$. In this base the matrix of Frobenius remains diagonal.

Proposition 5.2 *Let (D_i, φ_i) be filtered φ-modules with $[\varphi_i]_{\underline{\eta}^i} = diag(\vec{\alpha}_i, \vec{\delta}_i)$, $i = 1, 2$ and filtrations as in Proposition 5.1, with $\vec{x}^i = f_{J_{\vec{x}^i}}$ and $\vec{y}^i = f_{J_{\vec{y}^i}}$, $i = 1, 2$. The F-semisimple filtered φ-modules (D_1, φ_1) and (D_2, φ_2) are isomorphic if and only if either*

$$\left\{ \begin{array}{c} Nm_{\varphi}(\vec{\alpha}_1) = Nm_{\varphi}(\vec{\alpha}_2), \\ Nm_{\varphi}(\vec{\delta}_1) = Nm_{\varphi}(\vec{\delta}_2) \end{array} \right\}, \left\{ \begin{array}{c} I_0^+ \cap J_{\vec{x}^1} = I_0^+ \cap J_{\vec{x}^2}, \\ I_0^+ \cap J_{\vec{y}^1} = I_0^+ \cap J_{\vec{y}^2} \end{array} \right\}$$

and $f_{I_0^+ \cap J_{\vec{x}^1} \cap J_{\vec{y}^1}} \cdot \vec{A} = f_{I_0^+ \cap J_{\vec{x}^1} \cap J_{\vec{y}^1}} \cdot \vec{\Delta}$, viewed in the projective space $\mathbb{P}^{f-1}(E)$, where $\vec{A} = (1, \frac{\alpha_0^1}{\alpha_0^2}, \frac{\alpha_0^1 \alpha_1^1}{\alpha_0^2 \alpha_1^2}, ..., \frac{\alpha_0^1 \alpha_1^1 ... \alpha_{f-2}^1}{\alpha_0^2 \alpha_1^2 ... \alpha_{f-2}^2})$ and $\vec{\Delta} = (1, \frac{\delta_0^1}{\delta_0^2}, \frac{\delta_0^1 \delta_1^1}{\delta_0^2 \delta_1^2}, ..., \frac{\delta_0^1 \delta_1^1 ... \delta_{f-2}^1}{\delta_0^2 \delta_1^2 ... \delta_{f-2}^2})$, or

$$\left\{ \begin{array}{c} Nm_{\varphi}(\vec{\alpha}_1) = Nm_{\varphi}(\vec{\delta}_2), \\ Nm_{\varphi}(\vec{\delta}_1) = Nm_{\varphi}(\vec{\alpha}_2) \end{array} \right\}, \left\{ \begin{array}{c} I_0^+ \cap J_{\vec{x}^1} = I_0^+ \cap J_{\vec{y}^2}, \\ I_0^+ \cap J_{\vec{y}^1} = I_0^+ \cap J_{\vec{x}^2} \end{array} \right\}$$

and $f_{I_0^+ \cap J_{\vec{x}^1} \cap J_{\vec{y}^1}} \cdot \vec{B} = f_{I_0^+ \cap J_{\vec{x}^1} \cap J_{\vec{y}^1}} \cdot \vec{\Gamma}$ viewed in the projective space $\mathbb{P}^{f-1}(E)$, where $\vec{B} = (1, \frac{\delta_0^1}{\alpha_0^2}, \frac{\delta_0^1 \delta_1^1}{\alpha_0^2 \alpha_1^2}, ..., \frac{\delta_0^1 \delta_1^1 ... \delta_{f-2}^1}{\alpha_0^2 \alpha_1^2 ... \alpha_{f-2}^2})$ and $\vec{\Gamma} = (1, \frac{\alpha_0^1}{\delta_0^2}, \frac{\alpha_0^1 \alpha_1^1}{\delta_0^2 \delta_1^2}, ..., \frac{\alpha_0^1 \alpha_1^1 ... \alpha_{f-2}^1}{\delta_0^2 \delta_1^2 ... \delta_{f-2}^2})$. $\square$

The following corollary follows easily from Propositions 5.1 and 5.2.

Corollary 5.3 *Let (D, φ) be an F-semisimple, weakly admissible filtered φ-modules as in the beginning of this section.*
(I) If $Tr(\varphi^f) \in \mathcal{O}_E^{\times}$, then the corresponding crystalline representation is reducible;
(II) There exist infinitely many weakly admissible, non isomorphic, F-semisimple filtered φ-modules sharing the same characteristic polynomial and filtration with (D, φ) if and only if $\mid I_0^+ \cap J_{\vec{x}} \cap J_{\vec{y}} \mid > 1$. $\square$

The following Lemma allows us to compute determinants of two-dimensional crystalline representations in terms of their labeled Hodge-Tate weights.

Lemma 5.4 *If (D,φ) is a rank two weakly admissible filtered φ-module over K with E-coefficients and labeled Hodge-Tate weights $\{0,-k_i\}_{\tau_i}$, then $(\wedge^2_{E\otimes K}D, \wedge^2_{E\otimes K}\varphi)$ is weakly admissible with labeled Hodge-Tate weights $\{-k_i\}_{\tau_i}$.*

Proof. Let $\underline{\eta}=(\eta_1,\eta_2)$ be a standard base of (D,φ) and let $[\varphi]_{\underline{\eta}}$ and $\mathrm{Fil}^j D$ be as in Proposition 5.1. Clearly $(\wedge^2\varphi)(\eta_1\wedge\eta_2)=\vec{\alpha}\cdot\vec{\delta}(\eta_1\wedge\eta_2)$. Since $\mathrm{Fil}^j(D\wedge D)=\sum\limits_{j_1+j_2=j}(\mathrm{Fil}^{j_1}D\wedge_{E\otimes K}\mathrm{Fil}^{j_2}D)$ and $J_{\vec{x}}\cup J_{\vec{y}}=I_0$, a simple computation yields

$$\mathrm{Fil}^j(D\wedge D)=\begin{cases} E^{|\tau_{I_0}|}(\eta_1\wedge\eta_2) \ \text{ if } j\le w_0, \\ E^{|\tau_{I_1}|}(\eta_1\wedge\eta_2) \ \text{ if } \ 1+w_0\le j\le w_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ E^{|\tau_{I_{t-1}}|}(\eta_1\wedge\eta_2) \text{ if } 1+w_{t-2}\le j\le w_{t-1}, \\ 0 \ \text{ if } j\ge 1+w_{t-1}. \end{cases}$$

from which the statement about the labeled Hodge-Tate weights follows immediately. Weak admissibility is clear. ■

Corollary 5.5 *If V is the crystalline representation corresponding to (D,φ), then $(\det\bar{V})_{|\boldsymbol{I_K}}=\prod\limits_{i\in I_0}\omega_{f,\bar{\tau}_i}^{-k_i}$.*

Proof. Follows from Section 5.1. ■

5.3 Reducible two-dimensional crystalline representations of G_K modulo p

In this section, we compute the semisimplified mod p reductions of all reducible F-semisimple two-dimensional crystalline representations of G_K. The answer restricted on the inertia subgroup I_K turns out to depend only on the filtration of the corresponding weakly admissible filtered φ-module, when Frobenius is normalized as in Section 5.2.1. Let (D,φ) be as in Proposition 5.1 and assume that $v_p(Nm_\varphi(\vec{\alpha}))=\sum\limits_{\{i\in I_0:\ y_i=0\}}k_i$. Then (D,φ) corresponds to a reducible crystalline representation and $D_1=E^{|\tau|}\eta_1$ with $\varphi(\eta_1)=\vec{\alpha}\cdot\eta_1$ and

$$\mathrm{Fil}^j(D_1)=D_1\cap\mathrm{Fil}^jD=\begin{cases} D_1 & \text{if } \ j\le 0, \\ E^{|\tau_{I_{0,\vec{y}}}|}\eta_1 & \text{if } \ 1\le j\le w_0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ E^{|\tau_{I_{t-1,\vec{y}}}|}\eta_1 & \text{if } \ 1+w_{t-2}\le j\le w_{t-1}, \\ 0 & \text{if } \ j\ge 1+w_{t-1} \end{cases}$$

where $I_{r,\vec{y}}=I_r\cap J'_{\vec{y}}=\{i\in I_r: y_i=0\}$ is a weakly admissible submodule of (D,φ) (see [12, Prop. 2.10]). For each $i\in I_0$, let $r=r(i)$ be the largest index in $\{0,1,...,t-1\}$ such that $i\in I_r$. Notice that

$$e_i\mathrm{Fil}^j(D_1) = \begin{cases} e_iE^{|\tau|}\eta_1 & \text{if } j \leq m_i, \\ 0 & \text{if } j \geq 1+m_i, \end{cases} \quad \text{with } m_i = \begin{cases} 0 & \text{if } y_i \neq 0 \ , \\ w_r & \text{if } y_i = 0, \end{cases}$$

therefore (D_1, φ_1) corresponds to the effective crystalline character (with the notation of Section 5.1) $\chi_{C,\vec{0}} \cdot \chi_{e_0}^{m_0} \cdot \chi_{e_1}^{m_1} \cdot \ldots \cdot \chi_{e_{f-1}}^{m_{f-1}}$, where $C = \prod\limits_{i \in I_0} \alpha_i \cdot p^{-\sum\limits_{i\in I_0} k_i}$ and $\vec{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_{f-1})$. Let V be the crystalline representation corresponding to (D, φ). By Corollary 5.5, $(\det \bar{V})_{|\boldsymbol{I_K}} = \prod\limits_{i\in I_0} \omega_{f,\bar{\tau}_i}^{-k_i}$. Since $\bar{V}$ is semisimple,

$$\bar{V}_{|\mathbf{I}_K} = \begin{pmatrix} \prod\limits_{i\in I_0} \omega_{f,\bar{\tau}_i}^{-m_i} & 0 \\ 0 & \prod\limits_{i\in I_0} \omega_{f,\bar{\tau}_i}^{m_i-k_i} \end{pmatrix}$$

Similarly, if (D, φ) is as in Proposition 5.1 and $v_p(Nm_\varphi(\vec{\delta})) = \sum\limits_{\{i\in I_0:\ x_i=0\}} k_i$, then (D, φ) corresponds to a reducible crystalline representation V and

$$\bar{V}_{|\mathbf{I}_K} = \begin{pmatrix} \prod\limits_{i\in I_0} \omega_{f,\bar{\tau}_i}^{n_i-k_i} & 0 \\ 0 & \prod\limits_{i\in I_0} \omega_{f,\bar{\tau}_i}^{-n_i} \end{pmatrix}$$

where, for each $i \in I_0$, $r = r(i)$ is defined as above and $n_i = \begin{cases} 0 & \text{if } x_i \neq 0, \\ w_r & \text{if } x_i = 0. \end{cases}$

Theorem 5.6 *Let (D, φ) be a weakly admissible F-semisimple filtered φ-module in the standard form of Proposition 5.1, corresponding to a reducible representation. If $v_p(Nm_\varphi(\vec{\alpha})) = \sum\limits_{\{i\in I_0:\ y_i=0\}} k_i$, then*

$$\bar{V}_{|\mathbf{I}_K} = \begin{pmatrix} \prod\limits_{i\in I_0} \omega_{f,\bar{\tau}_i}^{-m_i} & 0 \\ 0 & \prod\limits_{i\in I_0} \omega_{f,\bar{\tau}_i}^{m_i-k_i} \end{pmatrix}$$

If $v_p(Nm_\varphi(\vec{\delta})) = \sum\limits_{\{i\in I_0:\ x_i=0\}} k_i$, then

$$\bar{V}_{|\mathbf{I}_K} = \begin{pmatrix} \prod\limits_{i\in I_0} \omega_{f,\bar{\tau}_i}^{n_i-k_i} & 0 \\ 0 & \prod\limits_{i\in I_0} \omega_{f,\bar{\tau}_i}^{-n_i} \end{pmatrix} \quad \square$$

Remark 5.7 *If both equalities hold then V is split-reducible and the semisimplified* mod p *reduction is given by either formula, since $I_0^+ \cap J_{\vec{x}} \cap J_{\vec{y}} = \varnothing$. For the results of this paragraph F-semisimplicity is irrelevant and the reductions of a non-F-semisimple reducible representation of G_K are given by the same formulas (see comments after Proposition 5.1). If (D, φ) is scalar, $\bar{V}_{|\mathbf{I}_K}$ is the sum of two trivial characters.*

5.4 Reductions of families consisting mostly of irreducible two-dimensional crystalline representations

Throughout this section the vectors $\vec{\imath} \in \{1,2,3,4\}^f$ are as in Section 4.2. We compute the semisimplified mod p reduction of some of the representations constructed in the same section.

The $[K:\mathbb{Q}_p]$ **even case** In the case where $\vec{\alpha}=\vec{0}$ and $[K:\mathbb{Q}_p]$ is even, the filtered modules $D^{\vec{\imath}}_{\vec{w},\vec{0}}$ are weakly admissible regardless of the choice of the vector $\vec{i}$ and the corresponding crystalline representations $V^{\vec{\imath}}_{\vec{w},\vec{0}}$ are split-reducible. Let

$$\Pi^{\vec{\imath}}(\vec{0})=\begin{pmatrix}0 & \beta_1\\ \gamma_1 & 0\end{pmatrix}\times\begin{pmatrix}0 & \beta_2\\ \gamma_2 & 0\end{pmatrix}\times ... \times\begin{pmatrix}0 & \beta_{f-1}\\ \gamma_{f-1} & 0\end{pmatrix}\times\begin{pmatrix}0 & \beta_0\\ \gamma_0 & 0\end{pmatrix}$$

with $\{\beta_i,\gamma_i\}=\{-1,C_iq^{k_i}\}$ and assume for simplicity that $C_i=1$ for all i.

Proposition 5.8 *If* $[K:\mathbb{Q}_p]=2s$, *then* $\mathbf{N}^{\vec{i}}_{\vec{w}}(\vec{0})=\mathbf{N}_1\oplus\mathbf{N}_2$, *where* $\mathbf{N}_i=(\mathcal{O}_E^{|\tau|})\zeta_i$ *with* $\varphi(\zeta_1)=(\beta_1,\gamma_2,\beta_3,\gamma_4,...,\beta_{f-1},\gamma_0)\zeta_1$ *and* $\varphi(\zeta_2)=(\gamma_1,\beta_2,\gamma_3,\beta_4,...,\gamma_{f-1},\beta_0)\zeta_2$.

Proof. Let $Q_i=Id$, $i=0,2,...,2s-2$, $Q_i=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ for $i=1,3,...,2s-1$ and $Q=Q_0\times Q_1\times ...\times Q_{2s-1}$. Let $\underline{\zeta}$ be the ordered base of $\mathbf{N}^{\vec{i}}_{\vec{w}}(\vec{0})$ defined by $Q=[1]^{\underline{\zeta}}_{\underline{\eta}}$. Then $(\eta_1,\eta_2)=(\zeta_1,\zeta_2)Q$, $[\varphi]_{\underline{\zeta}}=Q[\varphi]_{\underline{\eta}}\varphi(Q)^{-1}$ and $[\varphi]_{\underline{\zeta}}=\mathrm{diag}((\beta_1,\gamma_2,\beta_3,\gamma_4,...,\beta_{2s-1},\gamma_{2s}),(\gamma_1,\beta_2,\gamma_3,\beta_4,...,\gamma_{2s-1},\beta_{2s}))$. ■

Corollary 5.9 *Assume that* $[K:\mathbb{Q}_p]$ *is even. For any* $\vec{\imath}\in\{1,2,3,4\}^f$ *the filtered* φ*-modules* $D^{\vec{\imath}}_{\vec{w},\vec{0}}$ *are weakly admissible and the corresponding crystalline representations* $V^{\vec{\imath}}_{\vec{w},\vec{0}}$ *are split-reducible.* □

Recall the notation of Section

$$r_{\beta_i}=\begin{cases}k_i \text{ if } \beta_i=C_iq^{k_i},\\ 0 \text{ if } \beta_i=-1\end{cases}\quad\text{and } r_{\gamma_i}=\begin{cases}k_i \text{ if } \gamma_i=C_iq^{k_i},\\ 0 \text{ if } \gamma_i=-1\end{cases}$$

and let $r_1=\mid\{j\in\{1,3,...,2s-1\}:i_j\in\{1,2\}\}\mid+\mid\{j\in\{0,2,...,2s\}:i_j\in\{3,4\}\}\mid$. We have the following

Corollary 5.10 *If* $f=[K:\mathbb{Q}_p]$ *is even, then for all* $\vec{\imath}\in\{1,2,3,4\}^f$

$$\bar{V}^{\vec{\imath}}_{\vec{w},\vec{0}}\simeq\begin{pmatrix}\prod_{i\in\{0,2,...,f-2\}}\bar{\chi}_{e_i}^{r_{\beta_{i+1}}}\prod_{i\in\{1,3,...,f-1\}}\bar{\chi}_{e_i}^{r_{\gamma_{i+1}}}\mu & 0\\ 0 & \prod_{i\in\{0,2,...,f-2\}}\bar{\chi}_{e_i}^{r_{\gamma_{i+1}}}\prod_{i\in\{1,3,...,f-1\}}\bar{\chi}_{e_i}^{r_{\beta_{i+1}}}\mu\end{pmatrix}$$

where $\mu=\mu_{\sqrt[f]{(-1)^{r_1}}}$ *is the unramified character which maps* $Frob_K$ *to* $\sqrt[f]{(-1)^{r_1}}$ *and*

$$(\bar{V}^{\vec{\imath}}_{\vec{w},\vec{0}})_{|\mathrm{I}_K}\simeq\begin{pmatrix}\prod_{i\in\{1,3,...,f-1\}}\omega_{f,\bar{\tau}_i}^{-r_{\beta_i}}\prod_{i\in\{0,2,...,f-2\}}\omega_{f,\bar{\tau}_i}^{-r_{\gamma_i}} & 0\\ 0 & \prod_{i\in\{1,3,...,f-1\}}\omega_{f,\bar{\tau}_i}^{-r_{\gamma_i}}\prod_{i\in\{0,2,...,f-2\}}\omega_{f,\bar{\tau}_i}^{-r_{\beta_i}}\end{pmatrix}$$

where $\omega_{f,\bar{\tau}_i}$ *is as in Section 5.1.*

Proof. By Corollary 5.9, $V^{\vec{i}}_{\vec{w},\vec{\alpha}} = W_1 \oplus W_2$, where W_i is the one-dimensional E-linear crystalline representation of G_K corresponding to the Wach module $\mathbf{N}_i$. The $\bar{W}_i$ and $(\bar{W}_i)\mid_{\boldsymbol{I_K}}$ have been computed in Section 5.1. ■

The formulas of the previous Corollary remain the same without semisimplifying. The isomorphism class of a two-dimensional $\bar{\mathbb{F}}_p$-linear semisimple representation of G_K of given determinant is completely determined by its restriction on I_K.

Corollary 5.11 *Assume $[K : \mathbb{Q}_p]$ is even.*
(I) If the labeled Hodge-Tate weights are all equal and either $(i_j, i_{j+1}) \in \mathcal{E}_1$ for all $j \in \{1, 3, 5, ..., f-1\}$ or $(i_j, i_{j+1}) \in \mathcal{E}_2$ for all $j \in \{1, 3, 5, ..., f-1\}$. If $v_p(\vec{\alpha}) > \lfloor \frac{k}{p-1} \rfloor \cdot \vec{1}$, then $\bar{V}^{\vec{i}}_{\vec{w},\vec{\alpha}} \simeq \bar{V}^{\vec{i}}_{\vec{w},\vec{0}}$.
(II) If the labeled weights are not all equal, or if they are equal and we are not in case (I), then (i) If $v_p(\vec{\alpha}) > \lfloor \frac{k-1}{p-1} \rfloor \cdot \vec{1}$, then $\bar{V}^{\vec{i}}_{\vec{w},\vec{\alpha}} \simeq \bar{V}^{\vec{i}}_{\vec{w},\vec{0}}$; (ii) If all the i_t are equal and $k_i = p$ for all i, then $\bar{V}^{\vec{i}}_{\vec{w},\vec{\alpha}} \simeq \bar{V}^{\vec{i}}_{\vec{w},\vec{0}}$ for any $\vec{\alpha} \in m_E^f$.

Proof. Follows from Theorem 3.7, where for (II) (iv) we take into account that the exponents m_ℓ defined in Lemma 4.3 are not the "best possible". For example, if $\Pi_i = \begin{pmatrix} 0 & -1 \\ C_i q^{k_i} & X_i z_i \end{pmatrix}$ and $k_i = p$ for all $i \in I_0$, we may take $m_\ell = 0$ independently of the parity of f. To prove this, notice that if $\alpha = \sum_{n=0}^{\infty} \alpha_n \pi^n \in \mathbb{Q}_p[[\pi]]$ and $v_p(a_i) \geq 0$ for all $i = 0, 1, ..., p-2$ and $v_p(a_{p-1}) \geq -1$, then the first $p-1$ coefficients of α^p are in $\mathbb{Z}_p$. This is easy to check using the binomial expansion. The remark follows by the proof of Lemma 4.3. Similarly when all the Π_i are of type 2,3 or 4. ■

The $[K : \mathbb{Q}_p]$ odd case In this case, we use the Wach module associated to an effective two-dimensional crystalline representation V to exhibit rank one sub-(φ, Γ)-modules of the modulo p étale (φ, Γ)-modules $D(\bar{V})$, and therefore sub representations of $\bar{V}$. To do so, it is necessary to assume some divisibility condition involving the labeled Hodge-Tate weights of V, which guarantees that $\bar{V}$ is reducible.

Proposition 5.12 *If $[K : \mathbb{Q}_p]$ is odd, the representations $V^{\vec{\imath}}_{\vec{w},\vec{0}}$ are irreducible for any $\vec{\imath} \in \{1, 2, 3, 4\}^f$.*

Proof. If $[\varphi^f]_{\underline{\eta}} = A_0 \times A_1 \times ... \times A_{f-1}$, the matrices A_i are all conjugate and $A_i = \begin{pmatrix} 0 & b_i \\ c_i & 0 \end{pmatrix}$ with $b_i = \prod_{j=1}^{s+1} \beta_{i+2j-1} \prod_{j=1}^{s} \gamma_{i+2j}$ and $c_i = \prod_{j=1}^{s+1} \gamma_{i+2j-1} \prod_{j=1}^{s} \beta_{i+2j}$. Let $\varepsilon_0 \neq \varepsilon_1$ be the eigenvalues of the A_i and $Q_i = \begin{pmatrix} a_i & \varepsilon_0 \\ \varepsilon_1 & d_i \end{pmatrix}$ with $a_i = \prod_{j=1}^{s+1} \gamma_{i+2j-1} \prod_{j=1}^{s} \beta_{i+2j}$ and $d_i = \prod_{j=1}^{s+1} \beta_{i+2j-1} \prod_{j=1}^{s} \gamma_{i+2j}$. Then $Q_i A_i Q_i^{-1} = \mathrm{diag}(\varepsilon_0, \varepsilon_1)$ and $Q = Q_0 \times Q_1 \times ... \times Q_{f-1} = \begin{pmatrix} \vec{Q}_{11} & \vec{Q}_{12} \\ \vec{Q}_{21} & \vec{Q}_{22} \end{pmatrix}$. Let $\underline{\zeta}$ be the base defined by the matrix Q so that $Q = [1]^{\underline{\zeta}}_{\underline{\eta}}$. Then $(\eta_1, \eta_2) = (\vec{x} \cdot \vec{Q}_{11} + \vec{y} \cdot \vec{Q}_{12})\zeta_1 + (\vec{x} \cdot \vec{Q}_{21} + \vec{y} \cdot \vec{Q}_{22})\zeta_2$. If $\vec{z} = \vec{x} \cdot \vec{Q}_{11} + \vec{y} \cdot \vec{Q}_{12}$ and $\vec{u} = \vec{x} \cdot \vec{Q}_{21} + \vec{y} \cdot \vec{Q}_{22}$, one can easily check that $z_i u_i \neq 0$ for all i. Let $[\varphi]_{\underline{e}} = \mathrm{diag}(\vec{A}, \vec{\Delta})$, then $f \cdot v_p(Nm_\varphi(\vec{A})) = v_p(\varepsilon_0)$ and $f \cdot v_p(Nm_\varphi(\vec{\Delta})) = v_p(\varepsilon_1)$. Since $v_p(\varepsilon_0), v_p(\varepsilon_1) > 0$ and $z_i u_i \neq 0$ for all i, the proposition follows from Proposition 5.1. ■

Let $\bar{\Pi}^{\vec{i}}(\vec{0}) = \Pi_1(0) \times \Pi_2(0) \times ... \times \Pi_{f-1}(0) \times \Pi_0(0) \bmod m_E$ with $\bar{\Pi}_j(0) \bmod m_E = \begin{pmatrix} 0 & \beta_j \\ \gamma_j & 0 \end{pmatrix}$ where

$$\beta_j = \begin{cases} -1 & \text{if } i_j \in \{1,2\}, \\ \pi^{(p-1)k_j} & \text{if } i_j \in \{3,4\} \end{cases} \quad \text{and} \quad \gamma_j = \begin{cases} \pi^{(p-1)k_j} & \text{if } i_j \in \{1,2\}, \\ -1 & \text{if } i_j \in \{3,4\}. \end{cases}$$

We abuse notation and write $\bar{\Pi}^{\vec{i}}(\vec{0}) \bmod m_E = \begin{pmatrix} \vec{0} & \vec{\beta} \\ \vec{\gamma} & \vec{0} \end{pmatrix}$. We equip $\bar{N}_{C,\vec{w}} = (k_E[[\pi]]^{|\tau|})\eta_1 \oplus (k_E[[\pi]]^{|\tau|})\eta_2$ with the Frobenius endomorphism defined by $(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2)\bar{\Pi}^{\vec{i}}(\vec{0})$ and search for elements $\vec{\delta} = \vec{A}\eta_1 + \vec{B}\eta_2 \in k_E((\pi)))^{|\tau|} \bigotimes_{k_E[[\pi]]^{|\tau|}} \bar{N}_{C,\vec{w}}$ such that $\varphi(\vec{A}\eta_1 + \vec{B}\eta_2) = \lambda(\vec{A}\eta_1 + \vec{B}\eta_2)$ (1) for some $\lambda \in k_E^{\times}$. Let $\vec{A} = (A_0, A_1, ..., A_{f-1})$, (1) is equivalent to $\varphi^2(A_{i+2}) = \lambda^2 A_i \beta_i \varphi(\gamma_{i+1})^{-1}$ for all $i \in I_0$ and $\vec{B} = \lambda^{-1}\varphi(\vec{A})\vec{\gamma}$. Let $f = 2s+1$. A computation gives

$$\begin{aligned} &\varphi^{4s+2}(A_0) = A_0\lambda^{4s+2}\beta_0\varphi^2(\beta_2)...\varphi^{2s}(\beta_{2s})\varphi^{2s+2}(\beta_1)\varphi^{2s+4}(\beta_3)...\varphi^{4s}(\beta_{2s-1})\times \\ &\varphi(\gamma_1)^{-1}\varphi^3(\gamma_3)^{-1}...\varphi^{2s-1}(\gamma_{2s-1})^{-1}\varphi^{2s+1}(\gamma_0)^{-1}\varphi^{2s+3}(\gamma_2)^{-1}...\varphi^{4s+1}(\gamma_{2s})^{-1} \end{aligned} \tag{2}$$

For $i \in I_0$ we define

$$\ell_j^{\beta} = \begin{cases} 0 & \text{if } i_j \in \{1,2\}, \\ k_j(p-1) & \text{if } i_j \in \{3,4\} \end{cases} \quad \text{and } \ell_j^{\gamma} = \begin{cases} k_j(p-1) & \text{if } i_j \in \{1,2\}, \\ 0 & \text{if } i_j \in \{3,4\}. \end{cases}$$

Then (2) becomes $\varphi^{4s+2}(A_0) = A_0\lambda^{4s+2}(-1)^{2s+1}\pi^{\sum_{i=0}^{s} p^{2i}(\ell_{2i}^{\beta} - p^{2s+1}\ell_{2i}^{\gamma}) + \sum_{i=1}^{s} p^{2i-1}(p^{2s+1}\ell_{2i-1}^{\beta} - \ell_{2i-1}^{\gamma})}$.

Let $\lambda = \sqrt{-1}$. If $p^{4s+2} - 1 \mid \sum_{i=0}^{s} p^{2i}(\ell_{2i}^{\beta} - p^{2s+1}\ell_{2i}^{\gamma}) + \sum_{i=1}^{s} p^{2i-1}(p^{2s+1}\ell_{2i-1}^{\beta} - \ell_{2i-1}^{\gamma})$ and the quotient is r, then π^r is the unique (up to nonzero constant) solution of (2). The equations above imply that (1) has a unique solution up to a nonzero constant. We have $\varphi(\vec{\delta}) = \sqrt{-1}\vec{\delta}$. Let $\gamma \in G_K$ and $\gamma\vec{\delta} = \vec{\delta}'$. Arguing as in Section 5.1 we see that $\varphi(\vec{\delta}') = \sqrt{-1}\vec{\delta}'$ and $\gamma\vec{\delta}' = \omega(\gamma)^r\vec{\delta}'$. Since $\det \bar{V}_{\vec{w},\vec{0}}^{\vec{\iota}} = \prod_{i \in I_0} \bar{\chi}_{e_i}^{k_{i+1}}$ and since $\bar{V}_{\vec{w},\vec{0}}^{\vec{\iota}}$ is semisimple we have

$$\bar{V}_{\vec{w},\vec{0}}^{\vec{\iota}} = \begin{pmatrix} \mu_{\sqrt{-1}} \cdot \omega^r & 0 \\ 0 & \mu_{-\sqrt{-1}} \cdot \omega^{-r} \prod_{i \in I_0} \bar{\chi}_{e_i}^{k_{i+1}} \end{pmatrix}$$

and

$$(\bar{V}_{\vec{w},\vec{0}}^{\vec{\iota}})_{|\mathbf{I}_K} = \begin{pmatrix} \omega^r & 0 \\ 0 & \omega^{-r} \prod_{i \in I_0} \omega_{f,\bar{\tau}_i}^{-k_i} \end{pmatrix}$$

Let $r := \frac{\{\sum_{i=0}^{s} p^{2i}(\ell_{2i}^{\beta} - p^{2s+1}\ell_{2i}^{\gamma}) + \sum_{i=1}^{s} p^{2i-1}(p^{2s+1}\ell_{2i-1}^{\beta} - \ell_{2i-1}^{\gamma})\}}{p^{4s+2}-1}$. We have the following

Theorem 5.13 *If $[K : \mathbb{Q}_p] = 2s+1$, the for any $\vec{i} \in \{1,2,3,4\}^f$, the representations $V_{\vec{w},\vec{0}}^{\vec{\iota}}$ are irreducible. If $r \in \mathbb{N}$, then*

$$\bar{V}_{\vec{w},\vec{0}}^{\vec{\iota}} = \begin{pmatrix} \mu_{\sqrt{-1}} \cdot \omega^r & 0 \\ 0 & \mu_{-\sqrt{-1}} \cdot \omega^{-r} \prod_{i \in I_0} \bar{\chi}_{e_i}^{k_{i+1}} \end{pmatrix}$$

and

$$(\bar{V}^{\vec{\iota}}_{\vec{w},\vec{0}})_{|\mathbf{I}_K} = \begin{pmatrix} \omega^r & 0 \\ 0 & \omega^{-r} \prod\limits_{i \in I_0} \omega_{f,\bar{\tau}_i}^{-k_i} \end{pmatrix}$$

where ω is the mod p *cyclotomic character and $\omega_{f,\bar{\tau}_i}$ is as in Section 5.1.*

Corollary 5.14 *Assume $[K:\mathbb{Q}_p]$ is odd. The following isomorphisms hold:*
(i) If $v_p(\vec{\alpha}) > \lfloor \frac{k-1}{p-1} \rfloor \cdot \vec{1}$, then $\bar{V}^{\vec{i}}_{\vec{w},\vec{\alpha}} \simeq \bar{V}^{\vec{i}}_{\vec{w},\vec{0}}$ for any $\vec{i} \in \{1,2,3,4\}^f$;
(ii) If $\vec{i} \in \{\vec{1},\vec{2},\vec{3},\vec{4}\}$ and $\vec{w} = p \cdot \vec{1}$, then $\bar{V}^{\vec{i}}_{\vec{w},\vec{\alpha}} \simeq \bar{V}^{\vec{i}}_{\vec{w},\vec{0}}$ for any $\vec{\alpha} \in m_E^f$.

Proof. The same as for Corollary 5.11. ■
If $\vec{\iota} = \vec{1}$ and $\vec{a}, \vec{b} \in m_E^f$ we write $V_{\vec{w},\vec{a}}$ instead of $V^{\vec{1}}_{\vec{w},\vec{a}}$. One can easily prove that (i) If f is even, $V_{\vec{w},\vec{a}} \simeq V_{\vec{w},\vec{b}}$ if and only if there exists $\lambda \in E^\times$ such that $\lambda\,(a_0,a_2,...,a_{f-2}) = (b_0,b_2,...,b_{f-2})$ and $(a_1,a_3,...,a_{f-1}) = \lambda\,(b_1,b_3,...,b_{f-1})$. (ii) If f is odd, $V_{\vec{w},\vec{a}} \simeq V_{\vec{w},\vec{b}}$ if and only if $\vec{a} = \vec{b}$. Using Proposition 5.1 we see that for a generic $\vec{\alpha}$ and for certain choices of the vector $\vec{i}$ the representations $V^{\vec{i}}_{\vec{w},\vec{\alpha}}$ are "mostly irreducible". For example, when $f = 2$ the representations $V_{\vec{w},\vec{\alpha}}$ are irreducible if and only if $Nm_\varphi(\vec{\alpha}) \neq \vec{0}$. Their reductions mod p, nonetheless, are the same as those of the split-reducible representations $V_{\vec{w},\vec{0}}$, assuming that $v_p(\vec{\alpha})$ is suitably large.

We now study the filtered φ-modules $D^{\vec{i}}_{\vec{w},\vec{\alpha}}$ defined in section with $f = [K:\mathbb{Q}_p]$ even and vectors $\vec{\iota}$ such that $(i_j, i_{j+1}) \in \mathcal{C}_1$ for all $j \in \{1,3,5,...,f-1\}$ or $(i_j,i_{j+1}) \in \mathcal{C}_2$ for all $j \in \{1,3,5,...,f-1\}$ (see Definition 4.7). Assume that $\vec{\alpha}$ is any vector in $\mathcal{O}_E^f$. We may base change by the matrix $Q^f = Q \times Q \times ... \times Q$, where $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and only consider (up to isomorphism) those filtered φ-modules $D^{\vec{i}}_{\vec{w},\vec{\alpha}}$ with $(i_j,i_{j+1}) \in \mathcal{C}_1$ for all $j \in \{1,3,5,...,f-1\}$. We have the following

Proposition 5.15 *For any $\vec{a} \in \mathcal{O}_E^f$ the filtered φ-modules $(D^{\vec{i}}_{\vec{w},\vec{a}}, \varphi)$, with $\vec{i} \in \{1,2,3,4\}^f$ and $(i_j,i_{j+1}) \in \mathcal{C}_1$ for all $j \in \{1,3,5,...,f-1\}$ are weakly admissible and the corresponding crystalline representations $V^{\vec{i}}_{\vec{w},\vec{a}}$ are reducible.*

Proof. Let $[\varphi]_{\underline{\eta}} = P^{(i_1)} \times P^{(i_2)} \times ... \times P^{(i_{f-1})} \times P^{(i_0)}$ and $[\varphi^f]_{\underline{\eta}} = Nm_\varphi([\varphi]_{\underline{\eta}}) = A_0 \times A_1 \times ... \times A_{f-1}$ with $A_j = P^{(i_{j+1})} \cdot P^{(i_{j+2})} \cdot ... \cdot P^{(i_{j+f-1})} \cdot P^{(i_{j+f})}$ (with indices viewed mod f) for $j = 0,1,...,f-1$. We write $P^{(i_j)} = \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}$. We easily see (because $0 \in \{\alpha_j, \delta_j\}$) that $A_0 = \begin{pmatrix} \beta_1\beta_3...\beta_{f-1}\gamma_2\gamma_4...\gamma_0 & 0 \\ * & \gamma_1\gamma_3...\gamma_{f-1}\beta_2\beta_4...\beta_0 \end{pmatrix}$. There exist matrices

$$Q_{2j} = \begin{pmatrix} \beta_1\beta_3...\beta_{2j-1}\gamma_2\gamma_4...\gamma_{2j} & 0 \\ * & \gamma_1\gamma_3...\gamma_{2j-1}\beta_2\beta_4...\beta_{2j} \end{pmatrix}$$

for $j = 0,1,...,\frac{f}{2}-1$ and

$$Q_{2i-1} = \begin{pmatrix} 0 & \beta_1\beta_3...\beta_{2j-1}\gamma_2\gamma_4...\gamma_{2j-2} \\ \gamma_1\gamma_3...\gamma_{2j-1}\beta_2\beta_4...\beta_{2j-2} & * \end{pmatrix}$$

for $j = 1, ..., \frac{f}{2}$ (with empty products equal to 1) such that

$$Q[\varphi^f]_{\underline{\eta}}\varphi(Q^{-1}) = diag(\beta_1\beta_3...\beta_{f-1}\gamma_2\gamma_4...\gamma_0 \cdot \vec{1}, \gamma_1\gamma_3...\gamma_{f-1}\beta_2\beta_4...\beta_0 \cdot \vec{1}),$$

where $Q = Q_0 \times Q_1 \times ... \times Q_{f-1}$. Let $\underline{\zeta} = (\zeta_1, \zeta_2)$ be the ordered base of $D^{\vec{i}}_{\vec{w},\vec{\alpha}}$ defined by $\underline{\eta} = \underline{\zeta} \cdot Q$. If the filtration with respect to the base $\underline{\eta}$ is defined by $\vec{x}\eta_1 + \vec{y}\eta_2$ as in Proposition 5.1, the filtration with respect to the base $\underline{\zeta}$ is defined by $\vec{z}\zeta_1 + \vec{w}\zeta_2$ where $\vec{z} = (\vec{Q}_{11} \cdot \vec{x} + \vec{Q}_{12} \cdot \vec{y})$ and $\vec{w} = (\vec{Q}_{21} \cdot \vec{x} + \vec{Q}_{22} \cdot \vec{y})$, where $\vec{Q} = \begin{pmatrix} \vec{Q}_{11} & \vec{Q}_{12} \\ \vec{Q}_{21} & \vec{Q}_{22} \end{pmatrix}$. By the proof of Lemma 2.3 ([12] § 2.1), $[\varphi]_{\underline{\zeta}} = \begin{pmatrix} \vec{\alpha} & \vec{0} \\ \vec{0} & \vec{\delta} \end{pmatrix}$ with $Nm_\varphi(\vec{\alpha}) = \beta_1\beta_3...\beta_{f-1}\gamma_2\gamma_4...\gamma_0 \cdot \vec{1}$ and $Nm_\varphi(\vec{\delta}) = \gamma_1\gamma_3...\gamma_{f-1}\beta_2\beta_4...\beta_0 \cdot \vec{1}$. Since

$$v_p(\beta_{2j}\gamma_{2j-1}) = \begin{cases} k_{2j-1} & \text{if } (i_{2j-1}, i_{2j}) = (1,2), \\ 0 & \text{if } (i_{2j-1}, i_{2j}) = (4,2), \\ k_{2j} + k_{2j-1} & \text{if } (i_{2j-1}, i_{2j}) = (1,3), \\ k_{2j} & \text{if } (i_{2j-1}, i_{2j}) = (4,3), \end{cases}$$

we see that

$$v_p(Nm_\varphi(\vec{\delta})) = \sum_{\substack{j=1 \\ (i_{2j-1}, i_{2j}) \in \{(1,3),(4,3)\}}}^{\frac{f}{2}} k_{2j} + \sum_{\substack{j=1 \\ (i_{2j-1}, i_{2j}) \in \{(1,2),(1,3)\}}}^{\frac{f}{2}} k_{2j-1}.$$

and

$$v_p(Nm_\varphi(\vec{\alpha})) = \sum_{\substack{j=1 \\ (i_{2j-1}, i_{2j}) \in \{(1,2),(4,2)\}}}^{\frac{f}{2}} k_{2j} + \sum_{\substack{j=1 \\ (i_{2j-1}, i_{2j}) \in \{(4,2),(4,3)\}}}^{\frac{f}{2}} k_{2j-1},$$

hence $v_p(Nm_\varphi(\vec{\alpha})Nm_\varphi(\vec{\delta})) = \sum_{i \in I_0} k_i$. Notice that $\vec{z} = (x_0, y_1\beta_1, x_2\beta_1\gamma_2, y_3\beta_1\beta_3\gamma_2, ..., y_{f-1}\beta_1\beta_3...\beta_{f-1}$ $\gamma_2\gamma_4...\gamma_{f-2})$ and $\vec{w} = (*x_0 + y_0, x_1\gamma_1 + *y_1, *x_2 + \gamma_1\beta_2 y_2, ..., \gamma_1\gamma_3...\gamma_{f-1}\beta_2\beta_4...\beta_{f-2}x_{f-1} + y_{f-1}*)$. Since $\beta_j\gamma_j \neq 0$ for all j, $\sum_{\{i \in I_0 : z_i = 0\}} k_i = \sum_{\substack{j=1 \\ x_{2j}=0}}^{\frac{f}{2}} k_{2j} + \sum_{\substack{j=1 \\ y_{2j-1}=0}}^{\frac{f}{2}} k_{2j-1}$. But $x_{2j} = 0$ if and only if $i_{2j} = 0$ ($i_{2j} \neq 4$ because $(i_j, i_{j+1}) \in \mathcal{C}_1$ for all $j \in \{1, 3, 5, ..., f-1\}$) if and only if $(i_{2j-1}, i_{2j}) \in \{(1,3),(4,3)\}$ and similarly $y_{2j-1} = 0$ if and only if $i_{2j-1} = 1$ if and only if $(i_{2j-1}, i_{2j}) \in \{(1,2),(1,3)\}$. Hence

$$v_p(Nm_\varphi(\vec{\delta})) = \sum_{\{i \in I_0 : z_i = 0\}} k_i = \sum_{\substack{j=1 \\ (i_{2j-1}, i_{2j}) \in \{(1,3),(4,3)\}}}^{\frac{f}{2}} k_{2j} + \sum_{\substack{j=1 \\ (i_{2j-1}, i_{2j}) \in \{(1,2),(1,3)\}}}^{\frac{f}{2}} k_{2j-1}.$$

By Proposition 5.1, it suffices to prove that $v_p(Nm_\varphi(\vec{\alpha})) \geq \sum\limits_{\{i\in I_0 : w_i=0\}} k_i$, or equivalently that

$$\sum_{\substack{j=1 \\ (i_{2j-1},i_{2j})\in\{(1,2),(4,2)\}}}^{\frac{f}{2}} k_{2j} + \sum_{\substack{j=1 \\ (i_{2j-1},i_{2j})\in\{(4,2),(4,3)\}}}^{\frac{f}{2}} k_{2j-1} \geq$$

$$\sum_{\substack{j=1 \\ \text{with } (i_{2j-1},i_{2j})\in\{(1,2),(1,3),(4,2),(4,3)\} \\ \text{and } w_{2j}=0}}^{\frac{f}{2}} k_{2j} + \sum_{\substack{j=1 \\ \text{with } (i_{2j-1},i_{2j})\in\{(1,2),(1,3),(4,2),(4,3)\} \\ \text{and } w_{2j-1}=0}}^{\frac{f}{2}} k_{2j-1}.$$

We notice that if $(i_{2j-1}, i_{2j}) \in \{(1,2),(1,3)\}$, then $w_{2j-1} \neq 0$. Indeed, in this case, $(x_{2j-1}, y_{2j-1}) = (1,0)$ and $w_{2j-1} = \gamma_1\gamma_3...\gamma_{2j-1}\beta_2\beta_4...\beta_{2j-2} \neq 0$. Similarly, if $(i_{2j-1}, i_{2j}) \in \{(1,3),(4,3)\}$, then $w_{2j} \neq 0$. ■

6 Families of two-dimensional crystalline representations of $G_{\mathbb{Q}_{p^2}}$

Let V be a two-dimensional crystalline E-representation of $G_{\mathbb{Q}_{p^2}}$ with labeled Hodge-Tate weights $(\{0,-k_0\},\{0,-k_1\})$, with k_i positive integers. Let $\underline{\eta}$ be a standard base of the corresponding filtered φ-module (D,φ), so that $[\varphi]_{\underline{\eta}}$ =diag$(\vec{\alpha},\vec{\delta})$ for vectors $\vec{\alpha},\vec{\delta} \in E^\times \times E^\times$ and

$$\mathrm{Fil}^j(D) = \begin{cases} (E\times E)\eta_1 \oplus (E\times E)\eta_2 & \text{if } j \leq 0, \\ (E\times E)f_{I_0}(f_{J_{\vec{x}}}\eta_1 + f_{J_{\vec{y}}}\eta_2) & \text{if } 1 \leq j \leq w_0, \\ (E\times E)f_{I_1}(f_{J_{\vec{x}}}\eta_1 + f_{J_{\vec{y}}}\eta_2) & \text{if } 1+w_0 \leq j \leq w_1, \\ 0 \text{ if } j \geq 1+w_1. \end{cases}$$

Assume that (D,φ) is F-semisimple and non-scalar. By Proposition 5.1, one sees that irreducible representations can only correspond to filtered φ-modules with $(J_{\vec{x}}, J_{\vec{y}}) = (\{0,1\},\{1,1\})$, or $(J_{\vec{x}}, J_{\vec{y}}) = (\{1,1\},\{0,1\})$, or $(J_{\vec{x}}, J_{\vec{y}}) = (\{1,1\},\{1,1\})$. Suppose that the characteristic polynomial of φ^2 is $X^2 - AX + Cp^{k_0+k_1}$ with $C \in \mathcal{O}_E^\times$. We may twist the representation by some unramified character and assume that $C=1$. Let ε_0 and ε_1 be the roots of the characteristic polynomial and assume that $A^2 \neq 4p^{k_0+k_1}$ so that $\varepsilon_0 \neq \varepsilon_1$. One has $\varepsilon_0+\varepsilon_1 = A$, $\varepsilon_0\varepsilon_1 = p^{k_0+k_1}$, $Nm_\varphi(\vec{\alpha}) = (\varepsilon_0,\varepsilon_0)$ and $Nm_\varphi(\vec{\delta}) = (\varepsilon_1,\varepsilon_1)$. If V is irreducible, then $v_p(A) > 0$ by Corollary 5.3. The purpose of this section is to apply Theorem 3.4 to construct families of irreducible two-dimensional crystalline representations of $G_{\mathbb{Q}_{p^2}}$. There are 64 choices for pairs of matrices (T_i, T_j), where T_i and T_j are of the eight types of Definition 4.2. Irreducibility excludes certain choices of $\Pi(\vec{X})$, e.g. the case $\Pi_1(X_1)\times\Pi_0(X_0) = T_5 \times T_8$ is excluded, since in this case (assuming that the assumptions of Theorem 3.4 are satisfied) the trace of the Frobenius of the corresponding crystalline representation would be in $\mathcal{O}_E^\times$. However, choices like $\Pi^{(2,5)}(\vec{X}) = \Pi_1(X_1)\times\Pi_0(X_0) = T_2\times T_5$ are good candidates for yielding irreducible representations. Here, we only consider the cases $T_2\times T_5$, $T_2\times T_8$ and compare them with the family given by $T_1\times T_1$ which was constructed in Section 4.1. Throughout this section k_0, k_1 will be positive integers, $k = \max\{k_0,k_1\}$ and $m = \lfloor\frac{k-1}{p-1}\rfloor$.

6.1 An infinite family of non-isomorphic, irreducible two-dimensional crystalline representations of $G_{\mathbb{Q}_{p^2}}$ sharing the same characteristic polynomial and filtration

Let $\Pi(\vec{X}) = \Pi^{(2,5)}(\vec{X}) = \Pi_1(X_1)\times\Pi_0(X_0) = \begin{pmatrix} X_1\varphi(z_1) & -1 \\ q^{k_1} & 0 \end{pmatrix} \times \begin{pmatrix} q^{k_0} & 0 \\ X_0\varphi(z_0) & 1 \end{pmatrix}$ where $z_i \in \mathbb{Z}_p[\pi]$ are polynomials of degree $\le k-1$ to be defined shortly. Let $G_\gamma^{(k)} = \begin{pmatrix} (x_0^\gamma, x_1^\gamma) & (0,0) \\ (0,0) & (y_0^\gamma, y_1^\gamma) \end{pmatrix}$, then $\Pi(\vec{X})\varphi(G_\gamma^{(k)}) - G_\gamma^{(k)}\gamma(\Pi(\vec{X})) =$

$$\begin{pmatrix} ((\varphi(z_1x_1^\gamma) - x_0^\gamma\varphi(\gamma z_1))X_1, q^{k_0}\varphi(x_0^\gamma) - x_1^\gamma(\gamma q)^{k_0}) & (-\varphi(y_1^\gamma)+x_0^\gamma, 0) \\ (q^{k_1}\varphi(x_1^\gamma) - y_0^\gamma(\gamma q)^{k_1}, (\varphi(z_0x_0^\gamma) - y_1^\gamma\varphi(\gamma z_0))X_0) & (0\ ,\ \varphi(y_0^\gamma) - y_1^\gamma) \end{pmatrix}$$

We want $\varphi(z_1x_1^\gamma) = x_0^\gamma\varphi(\gamma z_1)$, $q^{k_0}\varphi(x_0^\gamma) = x_1^\gamma(\gamma q)^{k_0}$, $\varphi(y_0^\gamma) = y_1^\gamma$, $\varphi(y_1^\gamma) = x_0^\gamma$ and $(\varphi(z_1x_1^\gamma) - x_0^\gamma\varphi(\gamma z_1))X_1,\ (\varphi(z_0x_0^\gamma) - y_1^\gamma\varphi(\gamma z_0))X_0) \in \pi^k\mathcal{O}_E[[\pi, X_0, X_1, ..., X_{f-1}]]$. A simple computation using Lemma 2.3 yields $x_1^\gamma = (\lambda_{4,\gamma})^{k_0}(\varphi^3(\lambda_{4,\gamma}))^{k_1}$, $y_0^\gamma = (\lambda_{4,\gamma})^{k_1}(\varphi(\lambda_{4,\gamma}))^{k_0}$, $x_0^\gamma = \varphi(y_1^\gamma)$, $y_1^\gamma = \varphi(y_0^\gamma)$.We need to define z_1 so that $\varphi(z_1x_1^\gamma) - x_0^\gamma\varphi(\gamma z_1) \in \pi^k\mathbb{Z}_p[[\pi]]$ for all $\gamma \in \Gamma_K$. Since $x_0^\gamma = \varphi(y_1^\gamma)$ and $x_1^\gamma \in 1+\pi^k\mathbb{Z}_p[[\pi]]$ (see Corollary 4.3 part (2)) it suffices to have $z_1 - \frac{B}{\gamma B}\gamma z_1 \in \pi^k\mathbb{Z}_p[[\pi]]$ for all $\gamma \in \Gamma_K$, where $B = \frac{\varphi(\lambda_4)^{k_1}\varphi^2(\lambda_4)^{k_0}}{\lambda_4^{k_0}\varphi^3(\lambda_4)^{k_1}}$. Since $B \in \mathcal{R}$ (see Corollary 4.4) and $B \equiv 1 \bmod \pi$, the existence of z_1 follows from Corollary 4.4. The existence of z_0 is proven similarly and the existence of the matrix $G_\gamma^{(k)}$ such that $\Pi(\vec{X})\varphi(G_\gamma^{(k)}) - G_\gamma^{(k)}\gamma(\Pi(\vec{X})) \in (\pi^k,\pi^k)\mathcal{M}_2$ for all $\gamma \in \Gamma_K$, and $G_\gamma^{(k)} \equiv \vec{Id} \bmod \pi$ follows. Let $P_i = \Pi_i \bmod \pi$. Since $z_i \equiv p^m \bmod \pi$, where $m = \lfloor \frac{k-1}{p-1} \rfloor$, $P_1P_0 = \begin{pmatrix} X_1p^{m+k_0} - X_0p^m & -1 \\ p^{k_0+k_1} & 0 \end{pmatrix}$ and $P_1P_0 \bmod (p, X_0, X_1) = -\bar{E}_{12}$, hence condition (iv) of Theorem 3.4 is satisfied. If P_1P_0 has eigenvalues $\{x, \lambda x\}$ with $\lambda \in \mathcal{O}_E^\times$, then $(1+\lambda)x = p^{k_1} + X_1p^{m+k_0}$ and $\lambda x^2 = p^{k_0+k_1}$ which is absurd. Hence all four conditions of Theorem 3.4 are satisfied and for each $\gamma \in \Gamma_K$, there exists a unique matrix $G_\gamma(\vec{X}) \in \mathcal{M}_2$ such that $G_\gamma(\vec{X}) \equiv \vec{Id} \bmod \pi$ and $\Pi(\vec{X})\varphi(G_\gamma(\vec{X})) = G_\gamma(\vec{X})\gamma(\Pi(\vec{X}))$. For any $\vec{a} \in m_E^f$ the module $\boldsymbol{N}(\vec{a}) = \left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta_1 \oplus \left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta_2$ equipped with semilinear φ and Γ_K-actions defined by $(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2)\Pi(\vec{a})$ and $(\gamma(\eta_1), \gamma(\eta_2)) = (\eta_1, \eta_2)G_\gamma(\vec{a})$ is a Wach module corresponding to some lattice in a crystalline two-dimensional E-representation of $G_{\mathbb{Q}_{p^2}}$ with labeled Hodge-Tate weights $(\{0, -k_0\}, \{0, -k_1\})$. The corresponding weakly admissible filtered φ-module $(D_{\vec{w},\vec{a}}, \varphi)$ is defined by

$$(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2)\begin{pmatrix} (\alpha_1, p^{k_0}) & (-1, 0) \\ (p^{k_1}, \alpha_0) & (0, 1) \end{pmatrix} \text{ with } \alpha_i = a_ip^m$$

and

$$\mathrm{Fil}^j(D_{\vec{w},\vec{a}}^{(2,5)}) = \begin{cases} (E\times E)\eta_1 \oplus (E\times E)\eta_2 & \text{if } j \le 0, \\ (E\times E)f_{I_0}(f_{J_{\vec{x}}}\eta_1 + f_{J_{\vec{y}}}\eta_2) & \text{if } 1 \le j \le w_0, \\ (E\times E)f_{I_1}(f_{J_{\vec{x}}}\eta_1 + f_{J_{\vec{y}}}\eta_2) & \text{if } 1+w_0 \le j \le w_1, \\ 0 \text{ if } j \ge 1+w_1. \end{cases}$$

with $\vec{x} = (1,1)$ and $\vec{y} = (-\alpha_0, \alpha_1)$ (see Proposition 4.15). We change the base to have the matrix of Frobenius in the standard form of Proposition 5.1. First, we diagonalize $[\varphi^2]_{\underline{\eta}}$. We have $P_1(a_1)P_0(a_0) = \begin{pmatrix} \alpha_1p^{k_0} - \alpha_0 & -1 \\ p^{k_0+k_1} & 0 \end{pmatrix}$ and $P_0(a_0)P_1(a_1) = \begin{pmatrix} \alpha_1p^{k_0} & -p^{k_0} \\ \alpha_0\alpha_1 + p^{k_1} & -\alpha_0 \end{pmatrix}$ which

both have characteristic polynomial $X^2-(\alpha_1 p^{k_0}-\alpha_0)X+p^{k_0+k_1}$. Let $A=\alpha_1 p^{k_0}-\alpha_0$ be such that $A^2\neq 4p^{k_0+k_1}$ so that the roots $\varepsilon_0,\varepsilon_1$ of the characteristic polynomial be distinct. Then $Q_0P_1(a_1)P_0(a_0)Q_0^{-1}$ $=\mathrm{diag}(\varepsilon_0,\varepsilon_1)=Q_1P_0(a_0)P_1(a_1)Q_1^{-1}$, where $Q_0=\begin{pmatrix}\varepsilon_0 & -1\\ \varepsilon_1 & -1\end{pmatrix}$ and $Q_1=\begin{pmatrix}\varepsilon_0+\alpha_0 & -p^{k_0}\\ \varepsilon_1+\alpha_0 & -p^{k_0}\end{pmatrix}$. Let $Q=Q_0\times Q_1$ and consider the ordered base $\underline{\xi}=(\xi_1,\xi_2)$ of $D^{(2,5)}_{\vec{w},\vec{a}}$ defined by $(\eta_1,\eta_2)=(\xi_1,\xi_2)Q$. Then $[\varphi]_{\underline{\xi}}=Q[\varphi]_{\underline{\eta}}\varphi(Q)^{-1}=\begin{pmatrix}(\varepsilon_0p^{-k_0},p^{k_0}) & (0,0)\\ (0,0) & (\varepsilon_1p^{-k_0},p^{k_0})\end{pmatrix}$, $\eta_1=(\varepsilon_0,\varepsilon_0+\alpha_0)\xi_1+(\varepsilon_1,\varepsilon_1+\alpha_0)\xi_2$, $\eta_2=(-1,-p^{k_0})\xi_1+(-1,-p^{k_0})\xi_2$ and $\vec{x}\eta_1+\vec{y}\eta_2=(\varepsilon_0+\alpha_0,\varepsilon_0+\alpha_0-p^{k_0}\alpha_1)\xi_1+(\varepsilon_1+\alpha_0,\varepsilon_1+\alpha_0-\alpha_1p^{k_0})\xi_2$.
(I) If $\varepsilon_i+\alpha_0\neq 0$ for $i=1,2$ (notice that $\varepsilon_i+\alpha_0-p^{k_0}\alpha_1\neq 0$) we may further base-change by $\zeta_1=(\varepsilon_0+\alpha_0,\varepsilon_0+\alpha_0-p^{k_0}\alpha_1)\xi_1$ and $\zeta_2=(\varepsilon_1+\alpha_0,\varepsilon_1+\alpha_0-\alpha_1p^{k_0})\xi_2$ and get

$$[\varphi]_{\underline{\zeta}}=\begin{pmatrix}(\frac{-p^{k_1}}{\varepsilon_0+\alpha_0},\frac{p^{k_0}(\varepsilon_0+\alpha_0)}{\varepsilon_0+\alpha_0-p^{k_0}\alpha_1}) & (0,0)\\ (0,0) & (\frac{-p^{k_1}}{\varepsilon_1+\alpha_0},\frac{p^{k_0}(\varepsilon_1+\alpha_0)}{\varepsilon_1+\alpha_0-p^{k_0}\alpha_1})\end{pmatrix}$$

and

$$\mathrm{Fil}^j(D^{(2,5)}_{\vec{w},\vec{a}})=\begin{cases}(E\times E)\zeta_1\oplus(E\times E)\zeta_2 \text{ if } j\leq 0,\\ (E\times E)f_{I_0}(\zeta_1+\zeta_2) \text{ if } 1\leq j\leq w_0,\\ (E\times E)f_{I_1}(\zeta_1+\zeta_2) \text{ if } 1+w_0\leq j\leq w_1,\\ 0 \text{ if } j\geq 1+w_1.\end{cases}$$

Since $Nm_\varphi\left(\frac{-p^{k_1}}{\varepsilon_0+\alpha_0},\frac{p^{k_0}(\varepsilon_0+\alpha_0)}{\varepsilon_0+\alpha_0-p^{k_0}\alpha_1}\right)=(\varepsilon_0,\varepsilon_0)$, $Nm_\varphi\left(\frac{-p^{k_1}}{\varepsilon_1+\alpha_0},\frac{p^{k_0}(\varepsilon_1+\alpha_0)}{\varepsilon_1+\alpha_0-p^{k_0}\alpha_1}\right)=(\varepsilon_1,\varepsilon_1)$ and $v_p(\varepsilon_i)>0$ for $i=1,2$, by Proposition 5.1 $V^{(2,5)}_{\vec{w},\vec{\alpha}}$ is irreducible (compare with Corollary 5.9). By Theorem 3.7, $\bar{V}^{(2,5)}_{\vec{w},\vec{\alpha}}\simeq\bar{V}^{(2,5)}_{\vec{w},\vec{0}}$ whenever $v_p(\vec{\alpha})>m\cdot\vec{1}$

Proposition 6.1 *Let $k_0,k_1>0$, $k=\max\{k_0,k_1\}$ and $m=\lfloor\frac{k-1}{p-1}\rfloor$. Fix $A\in p^m m_E$ with $A^2\neq 4p^{k_0+k_1}$ so that the roots $\varepsilon_0,\varepsilon_1$ of $X^2-AX+p^{k_0+k_1}$ be distinct. For each $\alpha\in p^m m_E$ with $\alpha\neq\varepsilon_i^{-1}p^{k_i}$, $i=0,1$ we consider the family of filtered φ-module $D^A_{\vec{w}}(\alpha)$ defined by*

$$[\varphi]_{\underline{\eta}}=\begin{pmatrix}\left(\frac{-p^{k_1}}{\varepsilon_0+\alpha p^{k_0}-A},\frac{p^{k_0}\left(\varepsilon_0+\alpha p^{k_0}-A\right)}{\varepsilon_0-A}\right) & (0,0)\\ (0,0) & \left(\frac{-p^{k_1}}{\varepsilon_1+\alpha p^{k_0}-A},\frac{p^{k_0}\left(\varepsilon_1+\alpha p^{k_0}-A\right)}{\varepsilon_1-A}\right)\end{pmatrix}$$

and

$$Fil^j(D^A_{\vec{w}}(\alpha))=\begin{cases}(E\times E)\eta_1\oplus(E\times E)\eta_2 \text{ if } j\leq 0,\\ (E\times E)f_{I_0}(\eta_1+\eta_2) \text{ if } 1\leq j\leq w_0,\\ (E\times E)f_{I_1}(\eta_1+\eta_2) \text{ if } 1+w_0\leq j\leq w_1,\\ 0 \text{ if } j\geq 1+w_1.\end{cases}$$

The filtered φ-modules $D^A_{\vec{w}}(\alpha)$ are weakly admissible and correspond to irreducible crystalline representations $V^A_{\vec{w}}(\alpha)$ with labeled Hodge-Tate weights $(\{0,-k_0\},\{0,-k_1\})$ and characteristic polynomial $X^2-AX+p^{k_0+k_1}$. Moreover,
(I) For any $\alpha,\beta\in p^m m_E\setminus\{\varepsilon_0^{-1}p^{k_0},\varepsilon_1^{-1}p^{k_1}\}$, $V^A_{\vec{w}}(\alpha)\simeq V^A_{\vec{w}}(\beta)$ if and only if $\alpha=\beta$ and

(II) For any $A, \alpha \in p^m m_E$, $\bar{V}^A_{\vec{w}}(\alpha) \simeq \bar{V}^0_{\vec{w}}(0)$. If $(p+1)(p^2+1) \mid -p^2(k_1+pk_0)$ and the quotient is r, then

$$\left(\bar{V}^0_{\vec{w}}(0)\right)_{|\boldsymbol{I}_K} \simeq \begin{pmatrix} \omega^r & 0 \\ 0 & \omega^{-r} \prod\limits_{i\in\{0,1\}} \omega_{2,\bar{\tau}_i}^{-k_i} \end{pmatrix}$$

Proof. Clearly $V^A_{\vec{w}}(\alpha) = V^{(2,5)}_{\vec{w},(\alpha p^{k_0}-A,\alpha)}$. By Proposition 5.1, $V^A_{\vec{w}}(\alpha) \simeq V^A_{\vec{w}}(\beta)$ if and only if $\frac{\varepsilon_1+\alpha p^{k_0}-A}{\varepsilon_0+\alpha p^{k_0}-A} = \frac{\varepsilon_1+\beta p^{k_0}-A}{\varepsilon_0+\beta p^{k_0}-A}$. A simple computation shows that this is equivalent to $\alpha = \beta$. By the discussion preceding the Proposition, $\bar{V}^A_{\vec{w}}(\alpha) = \bar{V}^{(2,5)}_{\vec{w},(\alpha p^{k_0}-A,\alpha)} \simeq \bar{V}^{(2,5)}_{\vec{w},(0,0)} = \bar{V}^0_{\vec{w}}(0)$. The reduction is computed as in the proof of Theorem 5.13. ■

Remark 6.2 *The example of this section yields crystalline representations of $G_{\mathbb{Q}_{p^2}}$ with labeled Hodge-Tate weights $(\{0,-k_0\},\{0,-k_1\})$, $J_{\vec{x}} = J_{\vec{y}} = I_0$ and characteristic polynomial $X^2 - AX + p^{k_0+k_1}$ with $v_p(A) > m$. Fixing the characteristic polynomial and the filtration, the isomorphism class of the corresponding crystalline representation changes as the quotient $\frac{\varepsilon_1+\alpha p^{k_0}-A}{\varepsilon_0+\alpha p^{k_0}-A}$ varies. In order to construct all the crystalline representations as above, one has to use different types of matrices and get outside the range of quotients $\frac{\varepsilon_1+\alpha p^{k_0}-A}{\varepsilon_0+\alpha p^{k_0}-A}$ of this example. Such examples are given in sections 6.2 and 6.3.*

(II) If $\varepsilon_0 = -\alpha_0$. Since ε_0 is a root of $X^2 - (\alpha_1 p^{k_0} - \alpha_0)X + p^{k_0+k_1}$ this is equivalent to $\alpha_0\alpha_1 = -p^{k_0+k_1}$ which can be the case. Then $\vec{x}\eta_1 + \vec{y}\eta_2 = (0, \varepsilon_0+\alpha_0-p^{k_0}\alpha_1)\xi_1 + (\varepsilon_1+\alpha_0, \varepsilon_1+\alpha_0-\alpha_1 p^{k_0})\xi_2$ and after an obvious base-change we get

$$[\varphi]_{\underline{\zeta}} = \begin{pmatrix} \left(\frac{\varepsilon_0}{p^{k_0}}, p^{k_0}\right) & (0,0) \\ (0,0) & \left(\frac{-p^{k_1}}{\varepsilon_1+\alpha_0}, \frac{(\varepsilon_1+\alpha_0)p^{k_0}}{\varepsilon_1+\alpha_0-\alpha_1 p^{k_0}}\right) \end{pmatrix}$$

and

$$\mathrm{Fil}^j(D^{(2,5)}_{\vec{w},\vec{a}}) = \begin{cases} (E\times E)\zeta_1 \oplus (E\times E)\zeta_2 \text{ if } j \le 0, \\ (E\times E)f_{I_0}((0,1)\zeta_1+\zeta_2) \text{ if } 1 \le j \le w_0, \\ (E\times E)f_{I_1}((0,1)\zeta_1+\zeta_2) \text{ if } 1+w_0 \le j \le w_1, \\ 0 \text{ if } j \ge 1+w_1. \end{cases}$$

Since $\alpha_0\alpha_1 = -p^{k_1}$, $v_p(\alpha_0) < k_1$ and since $\varepsilon_0\varepsilon_1 = p^{k_0+k_1}$ and $\varepsilon_0 = -\alpha_0$, $v_p(\varepsilon_1) > k_0$. Hence the representation $V^{(2,5)}_{\vec{w},\vec{a}}$ is irreducible (by Proposition 5.1). By Proposition 5.2 this is the unique (up to isomorphism) two-dimensional crystalline E-representation of $G_{\mathbb{Q}_{p^2}}$ with characteristic polynomial $X^2 - (\alpha_1 p^{k_0} - \alpha_0)X + p^{k_0+k_1}$ such that $\alpha_0\alpha_1 = -p^{k_1}$, and filtration as above.
(III) Similarly we prove that

$$[\varphi]_{\underline{\zeta}} = \begin{pmatrix} \left(\frac{-p^{k_1}}{\varepsilon_0+\alpha_0}, \frac{(\varepsilon_0+\alpha_0)p^{k_0}}{\varepsilon_0+\alpha_0-p^{k_0}\alpha_1}\right) & (0,0) \\ (0,0) & \left(\frac{\varepsilon_1}{p^{k_0}}, p^{k_0}\right) \end{pmatrix}$$

with

$$\mathrm{Fil}^j(D^{(2,5)}_{\vec{w},\vec{a}}) = \begin{cases} (E\times E)\zeta_1 \oplus (E\times E)\zeta_2 \text{ if } j \le 0, \\ (E\times E)f_{I_0}(\zeta_1+(0,1)\zeta_2) \text{ if } 1 \le j \le w_0, \\ (E\times E)f_{I_1}(\zeta_1+(0,1)\zeta_2) \text{ if } 1+w_0 \le j \le w_1, \\ 0 \text{ if } j \ge 1+w_1. \end{cases}$$

corresponds to the unique irreducible two-dimensional crystalline E-representation of $G_{\mathbb{Q}_{p^2}}$ with characteristic polynomial $X^2-(\alpha_1 p^{k_0}-\alpha_0)X+p^{k_0+k_1}$ such that $\alpha_0\alpha_1=-p^{k_1}$, and filtration as above.

6.2 A family of irreducible crystalline representations disjoint from the previous one

We repeat the process of the previous example to the case where $\Pi^{(2,8)}(\vec{X})=\Pi(\vec{X})=\Pi_1(X_1)\times \Pi_0(X_0)=\begin{pmatrix} X_1\varphi(z_1) & -1 \\ q^{k_1} & 0 \end{pmatrix}\times\begin{pmatrix} 1 & 0 \\ X_0 z_0 & q^{k_0} \end{pmatrix}$. Let $G_\gamma^{(k)}=\begin{pmatrix} (x_0^\gamma,x_1^\gamma) & (0,0) \\ (0,0) & (y_0^\gamma,y_1^\gamma) \end{pmatrix}$ with

$$y_0^\gamma=(\frac{q}{\gamma q})^{k_1}\varphi^3(\lambda_{4,\gamma})^{k_0}\varphi^4(\lambda_{4,\gamma})^{k_1},\ y_1^\gamma=(\lambda_{4,\gamma})^{k_0}\varphi(\lambda_{4,\gamma})^{k_1},\ x_0^\gamma=\varphi(y_1^\gamma),\ x_1^\gamma=\varphi^2(y_1^\gamma).$$

Then

$$\Pi(\vec{X})\varphi(G_\gamma^{(k)})-G_\gamma^{(k)}\gamma(\Pi(\vec{X}))=\begin{pmatrix} ((\varphi(z_1x_1^\gamma)-x_0^\gamma\varphi(\gamma z_1))X_1,0) & (0,0) \\ (0,(\varphi(z_0x_0^\gamma)-y_1^\gamma\varphi(\gamma z_0))X_0) & (0,0) \end{pmatrix}$$

For z_i defined similarly as in the previous section, $\Pi(\vec{X})\varphi(G_\gamma^{(k)})-G_\gamma^{(k)}\gamma(\Pi(\vec{X}))\in(\pi^k,\pi^k)\mathcal{M}_2$ and $G_\gamma^{(k)}\equiv \vec{Id}\bmod \pi$. Let $P_i=\Pi_i \bmod \pi$, then $P_1P_0=\begin{pmatrix} X_1p^m-X_0p^m & -p^{k_0} \\ p^{k_1} & 0 \end{pmatrix}$, where $m=\lfloor\frac{k-1}{p-1}\rfloor$ and $P_1P_0 \bmod(p,X_0,X_1)=\bar{0}$, hence condition (iv) of Theorem 3.4 is satisfied. If P_1P_0 has eigenvalues $\{x,\lambda x\}$ with $\lambda\in\mathcal{O}_E^\times$, then $(1+\lambda)x=X_1p^m-X_0p^m$ and $\lambda x^2=p^{k_0+k_1}$ which is absurd. Hence all four conditions of Theorem 3.4 are satisfied and for each $\gamma\in\Gamma_K$ there exists unique matrix $G_\gamma(\vec{X})\in\mathcal{M}_2$ such that $G_\gamma(\vec{X})\equiv\vec{Id}\bmod\pi$ and $\Pi(\vec{X})\varphi(G_\gamma(\vec{X}))=G_\gamma(\vec{X})\gamma(\Pi(\vec{X}))$. For any $\vec{a}\in m_E^2$, the module $\boldsymbol{N}(\vec{a})=\left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta_1\oplus\left(\mathcal{O}_E[[\pi]]^{|\tau|}\right)\eta_2$ equipped with semilinear φ and Γ_K-actions defined by $(\varphi(\eta_1),\varphi(\eta_2))=(\eta_1,\eta_2)\Pi(\vec{a})$ and $(\gamma(\eta_1),\gamma(\eta_2))=(\eta_1,\eta_2)G_\gamma(\vec{a})$ is a Wach module corresponding to some lattice in a crystalline two-dimensional E-representation of $G_{\mathbb{Q}_{p^2}}$ with labeled Hodge-Tate weights $(\{0,-k_0\},\{0,-k_1\})$. The corresponding weakly admissible filtered φ-module $(D_{\vec{w},\vec{a}}^{(2,8)},\varphi)$ is defined by

$$(\varphi(\eta_1),\varphi(\eta_2))=(\eta_1,\eta_2)\begin{pmatrix} (\alpha_1,1) & (-1,0) \\ (p^{k_1},\alpha_0) & (0,p^{k_0}) \end{pmatrix},$$

where $\alpha_i=a_ip^m$ and

$$\mathrm{Fil}^j(D_{\vec{w},\vec{a}}^{(2,8)})=\begin{cases} (E\times E)\eta_1\oplus(E\times E)\eta_2 & \text{if } j\le 0, \\ (E\times E)f_{I_0}(f_{J_{\vec{x}}}\eta_1+f_{J_{\vec{y}}}\eta_2) & \text{if } 1\le j\le w_0, \\ (E\times E)f_{I_1}(f_{J_{\vec{x}}}\eta_1+f_{J_{\vec{y}}}\eta_2) & \text{if } 1+w_0\le j\le w_1, \\ 0 \text{ if } j\ge 1+w_1. \end{cases}$$

with $\vec{x}=(0,1)$ and $\vec{y}=(1,\alpha_1)$ (see Proposition 4.15). We put the Frobenius in the standard form of Proposition 5.1. First, we diagonalize $[\varphi^2]_{\underline{\eta}}$. We have $P_1(a_1)P_0(a_0)=\begin{pmatrix} \alpha_1-\alpha_0 & -p^{k_0} \\ p^{k_1} & 0 \end{pmatrix}$ and $P_0(a_0)P_1(a_1)=\begin{pmatrix} \alpha_1 & -1 \\ \alpha_0\alpha_1+p^{k_0+k_1} & -\alpha_0 \end{pmatrix}$ which both have characteristic polynomial $X^2-(\alpha_1-\alpha_0)X+p^{k_0+k_1}$. Assume that $A=\alpha_1-\alpha_0$ is such that $A^2\neq 4p^{k_0+k_1}$ and let $\varepsilon_0\neq\varepsilon_1$ be

the distinct roots of the characteristic polynomial. Then $Q_0P_1(a_1)P_0(a_0)Q_0^{-1}$ $=\mathrm{diag}(\varepsilon_0,\varepsilon_1)$ $=$ $Q_1P_0(a_0)P_1(a_1)Q_1^{-1}$ where $Q_0=\begin{pmatrix}\varepsilon_0 & -p^{k_0}\\ \varepsilon_1 & -p^{k_0}\end{pmatrix}$ and $Q_1=\begin{pmatrix}\varepsilon_0+\alpha_0 & -1\\ \varepsilon_1+\alpha_0 & -1\end{pmatrix}$. Let $Q=Q_0\times Q_1$ and consider the ordered base $\underline{\xi}=(\xi_1,\xi_2)$ of $D^{(2,8)}_{\vec{w},\vec{a}}$ defined by $(\eta_1,\eta_2)=(\xi_1,\xi_2)Q$. Then $[\varphi]_{\underline{\xi}}=Q[\varphi]_{\underline{\eta}}\varphi(Q)^{-1}=\begin{pmatrix}(\varepsilon_0,1) & (0,0)\\ (0,0) & (\varepsilon_1,1)\end{pmatrix}$ and $\vec{x}\eta_1+\vec{y}\eta_2=(-p^{k_0},\alpha_0-\alpha_1+\varepsilon_0)\xi_1+(-p^{k_0},\alpha_0-\alpha_1+\varepsilon_1)\xi_2$. Since $\varepsilon_i\neq\alpha_1-\alpha_0$, an obvious base-change yields

$$[\varphi]_{\underline{\zeta}}=\begin{pmatrix}\left(p^{k_1},\frac{-p^{k_1}}{\alpha_0-\alpha_1+\varepsilon_0}\right) & (0,0)\\ (0,0) & \left(p^{k_1},\frac{-p^{k_1}}{\alpha_0-\alpha_1+\varepsilon_1}\right)\end{pmatrix}$$

and

$$\mathrm{Fil}^j(D^{(2,8)}_{\vec{w},\vec{a}})=\begin{cases}(E\times E)\zeta_1\oplus(E\times E)\zeta_2 \text{ if } j\leq 0,\\ (E\times E)f_{I_0}(\zeta_1+\zeta_2) \text{ if } 1\leq j\leq w_0,\\ (E\times E)f_{I_1}(\zeta_1+\zeta_2) \text{ if } 1+w_0\leq j\leq w_1,\\ 0 \text{ if } j\geq 1+w_1.\end{cases}$$

Proposition 6.3 *There is no representation of the family constructed in this section isomorphic to any representation of the family constructed in the previous section.*

Proof. Follows immediately from Proposition 5.2, given that $\frac{\varepsilon_1+\alpha p^{k_0}-A}{\varepsilon_0+\alpha p^{k_0}-A}\neq 1$ since $\varepsilon_0\neq\varepsilon_1$. ∎

Proposition 6.4 *If $v_p(\vec{a})>m\cdot\vec{1}$, then $\bar{V}^{(2,8)}_{\vec{w},\vec{\alpha}}\simeq\bar{V}^{(2,8)}_{\vec{w},\vec{0}}$. If $(p+1)(p^2+1)\mid -pk_0+p^2k_1$ and the quotient is r, then*

$$\left(\bar{V}^{(2,8)}_{\vec{w},\vec{0}}\right)_{|\boldsymbol{I}_K}\simeq\begin{pmatrix}\omega^r & 0\\ 0 & \omega^{-r}\prod\limits_{i\in\{0,1\}}\omega_{2,\bar{\tau}_i}^{-k_i}\end{pmatrix}.$$

Proof. The isomorphism $\bar{V}^{(2,8)}_{\vec{w},\vec{\alpha}}\simeq\bar{V}^{(2,8)}_{\vec{w},\vec{0}}$ follows from Theorem 3.7. The reductions are computed as in the proof of Theorem 5.13. ∎

6.3 A family consisting mostly of irreducible crystalline representations with the same reduction as a split-reducible representation

The cases where $\Pi(\vec{X})=\Pi^{(1,1)}(\vec{X})=\Pi_1(X_1)\times\Pi_0(X_0)=\begin{pmatrix}(0,0) & (-1,-1)\\ (q^{k_1},q^{k_0}) & (X_1z_1,X_0z_0)\end{pmatrix}$. This family has been studied in Section 5.4. Let $\vec{a}\in m_E^2$ and $\alpha_i=p^m a_i$. The crystalline representations $V^{(1,1)}_{\vec{w},\vec{\alpha}}$ correspond to the weakly admissible filtered φ-modules $D^{(1,1)}_{\vec{w},\vec{a}}$ with Frobenius endomorphisms

$$(\varphi(\eta_1),\varphi(\eta_2))=(\eta_1,\eta_2)\begin{pmatrix}(0,0) & (-1,-1)\\ (p^{k_1},p^{k_0}) & (\alpha_1,\alpha_0)\end{pmatrix}$$

and filtrations

$$\mathrm{Fil}^jD^{(1,1)}_{\vec{w},\vec{a}}=\begin{cases}(E\times E)\eta_1\oplus(E\times E)\eta_2 \text{ if } j\leq 0,\\ (E\times E)f_{I_0}\eta_1 \text{ if } 1\leq j\leq w_0,\\ (E\times E)f_{I_1}\eta_1 \text{ if } 1+w_0\leq j\leq w_1,\\ 0 \text{ if } j\geq 1+w_1.\end{cases}$$

If $\alpha_0\alpha_1 \neq 0$, arguing as in the previous sections we prove that there exists an ordered base $\underline{\zeta} = (\zeta_1, \zeta_2)$ such that $[\varphi]_{\underline{\zeta}} = \begin{pmatrix} \vec{A} & \vec{0} \\ \vec{0} & \vec{\Delta} \end{pmatrix}$, where

$$\vec{A} = \left(\frac{(p^{k_0}+\varepsilon_0)(p^{k_1}+\varepsilon_1) - \alpha_0\alpha_1\varepsilon_0}{\alpha_0(\varepsilon_1 - \varepsilon_0)}, \frac{(p^{k_1}+\varepsilon_0)(p^{k_0}+\varepsilon_1) - \alpha_0\alpha_1\varepsilon_0}{\alpha_1(\varepsilon_1 - \varepsilon_0)} \right),$$

$$\vec{\Delta} = \left(\frac{(p^{k_0}+\varepsilon_1)(p^{k_1}+\varepsilon_0) - \alpha_0\alpha_1\varepsilon_1}{\alpha_0(\varepsilon_0 - \varepsilon_1)}, \frac{(p^{k_1}+\varepsilon_1)(p^{k_0}+\varepsilon_0) - \alpha_0\alpha_1\varepsilon_1}{\alpha_1(\varepsilon_0 - \varepsilon_1)} \right),$$

and

$$\mathrm{Fil}^j(D^{(1,1)}_{\vec{w},\vec{a}}) = \begin{cases} (E \times E)\zeta_1 \oplus (E \times E)\zeta_2 \text{ if } j \leq 0, \\ (E \times E) f_{I_0}(\zeta_1 + \zeta_2) \text{ if } 1 \leq j \leq w_0, \\ (E \times E) f_{I_1}(\zeta_1 + \zeta_2) \text{ if } 1 + w_0 \leq j \leq w_1, \\ 0 \text{ if } j \geq 1 + w_1. \end{cases}$$

Since $Nm_\varphi(\vec{A}) = (\varepsilon_0, \varepsilon_0)$, $Nm_\varphi(\vec{\Delta}) = (\varepsilon_1, \varepsilon_1)$ and $v_p(\varepsilon_0)$, $v_p(\varepsilon_1) > 0$ the representations $V^{(1,1)}_{\vec{w},\vec{\alpha}}$ are irreducible by Proposition 5.1. $V^{(1,1)}_{\vec{w},\vec{0}}$ is split-reducible (see Corollary 5.9). By Corollaries 5.10 and 5.11 we see that if $v_p(\vec{a}) > m \cdot \vec{1}$, then

$$\left(\bar{V}^{(1,1)}_{\vec{w},\vec{\alpha}}\right)_{|\mathbf{I_K}} \simeq \begin{pmatrix} \omega_{2,\bar{\tau}_1}^{-k_1} & 0 \\ 0 & \omega_{2,\bar{\tau}_0}^{-k_0} \end{pmatrix}.$$

Proposition 6.5 *For any $\vec{\alpha}, \vec{\beta} \in p^m m_E$, $V^{(1,1)}_{\vec{w},\vec{\alpha}} \not\simeq V^{(2,8)}_{\vec{w},\vec{\beta}}$.*

Proof. Suppose $V^{(1,1)}_{\vec{w},\vec{\alpha}} \simeq V^{(2,8)}_{\vec{w},\vec{\beta}}$ and let $X^2 - AX + p^{k_0+k_1}$ with $A^2 \neq 4p^{k_0+k_1}$ be their common characteristic polynomial. By Proposition 5.2, $\dfrac{(p^{k_0}+\varepsilon_0)(p^{k_1}+\varepsilon_1) - \beta_0\beta_1\varepsilon_0}{(p^{k_0}+\varepsilon_1)(p^{k_1}+\varepsilon_0) - \beta_0\beta_1\varepsilon_1} = -1$. Since $A = \beta_0\beta_1 - p^{k_0} - p^{k_1}$ and $A = \varepsilon_0 + \varepsilon_1$ this is equivalent to $A^2 = 4p^{k_0+k_1}$ contradiction. ∎

We now search for isomorphisms between members of the families $V^{(1,1)}_{\vec{w},\vec{\alpha}}$ and $V^A_{\vec{w}}(\alpha)$. We prove that, under a certain condition, such isomorphisms exist and we use that to compute the mod p reduction of the representations $V^A_{\vec{w}}(\alpha)$ when the divisibility condition of Proposition 6.1 (II) doesn't hold.

Proposition 6.6 *If $k_0 \neq k_1$ and there exist $\beta_0, \beta_1 \in p^m m_E$ such that $p^{k_0} - p^{k_1} = \beta_0\beta_1$, then $V^{-2p^{k_1}}_{\vec{w}}(\alpha) \simeq V^{(1,1)}_{\vec{w},\vec{\beta}}$ for any $\alpha \in p^m m_E \setminus \{\frac{\varepsilon_0}{p^{k_0}}, \frac{\varepsilon_1}{p^{k_1}}\}$, where $\vec{\beta} = (\beta_0, \beta_1)$.*

Proof. The representations $V^{-2p^{k_1}}_{\vec{w}}(\alpha)$, $V^{(1,1)}_{\vec{w},\vec{\beta}}$ have characteristic polynomials $X^2 + 2p^{k_1}X + p^{k_0+k_1}$ and $X^2 - (\beta_0\beta - p^{k_0} - p^{k_1})X + p^{k_0+k_1}$ respectively, so we need $A = -2p^{k_1} = \beta_0\beta - p^{k_0} - p^{k_1}$ which implies $p^{k_0} - p^{k_1} = \beta_0\beta_1$. The roots of the characteristic polynomial are distinct if and only if $k_0 \neq k_1$. By Proposition 5.2 the representations are isomorphic if and only if $\frac{\varepsilon_1 + \alpha p^{k_0} - A}{\varepsilon_0 + \alpha p^{k_0} - A} = \dfrac{(p^{k_0}+\varepsilon_0)(p^{k_1}+\varepsilon_1) - \beta_0\beta_1\varepsilon_0}{(p^{k_0}+\varepsilon_1)(p^{k_1}+\varepsilon_0) - \beta_0\beta_1\varepsilon_1}$. Since the left hand side of the equation equals $\frac{\varepsilon_0}{\varepsilon_1}$, this is equivalent to $\varepsilon_1(p^{k_0}+\varepsilon_0)(p^{k_1}+\varepsilon_1) = \varepsilon_0(p^{k_0}+\varepsilon_1)(p^{k_1}+\varepsilon_0)$. A simple computation (given that $\varepsilon_0 \neq \varepsilon_1$ and $A = \varepsilon_0 + \varepsilon_1$) shows that this is equivalent to $A = -2p^{k_1}$, which is the case. ∎

Corollary 6.7 *If $k_0 \neq k_1$ and there exists $\beta_0, \beta_1 \in p^m m_E$ such that $p^{k_0} - p^{k_1} = \beta_0\beta_1$, then*

$$\left(\bar{V}^A_{\vec{w}}(\alpha)\right)_{|_{I_K}} \simeq \begin{pmatrix} \omega_{2,\bar{\tau}_1}^{-k_1} & 0 \\ 0 & \omega_{2,\bar{\tau}_0}^{-k_0} \end{pmatrix}$$

for any $A, \alpha \in p^m m_E$.

Proof. By Proposition 6.6, $V_{\vec{w}}^{-2p^{k_1}}(\alpha) \simeq V_{\vec{w},\vec{\beta}}^{(1,1)}$ for any $\alpha \in p^m m_E \setminus \{\frac{\varepsilon_0}{p^{k_0}}, \frac{\varepsilon_1}{p^{k_1}}\}$. Since $k_1 > m$ and $v_p(\alpha) > m$, $\bar{V}_{\vec{w}}^{-2p^{k_1}}(\alpha) \simeq \bar{V}^A_{\vec{w}}(\alpha) \simeq \bar{V}_{\vec{w},\vec{\beta}}^{(1,1)}$ and the corollary follows from Corollary 5.10. ■

7 Families of three-dimensional crystalline representations

In this section we apply Theorem 3.4 to construct a family of Wach modules corresponding to three-dimensional crystalline representations and compute the semisimplified modulo p reduction of the family. If V is a three-dimensional crystalline E-representation of $G_{\mathbb{Q}_p}$, we may twist by some power of the cyclotomic character and assume that the Hodge-Tate weights are $\{0, -k_1, -k_2\}$, with $k_i \geq 0$. For simplicity, we assume that the k_i are both positive. Let

$$\Pi(\vec{X}) = \Pi(X_0, X_1, X_2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & q^{k_1} & X_1 z_1 \\ q^{k_2} & X_0 z_0 & X_2 z_2 \end{pmatrix}$$

where $z_i \in \mathbb{Z}_p[\pi]$ are polynomials of degree $\leq k-1$ to be defined shortly, where $k = k_1 + k_2$. Let $G_\gamma^{(k)}$ =diag$(x_0^\gamma, x_1^\gamma, x_2^\gamma)$, then $\Pi(\vec{X})\varphi(G_\gamma^{(k)}) - G_\gamma^{(k)}\gamma(\Pi(\vec{X})) =$

$$\begin{pmatrix} 0 & 0 & x_0^\gamma - \varphi(x_2^\gamma) \\ 0 & q^{k_1}\varphi(x_1^\gamma) - x_1^\gamma(\gamma q)^{k_1} & \left(z_1\varphi(x_2^\gamma) - x_1^\gamma(\gamma z_1)\right) X_1 \\ q^{k_2}\varphi(x_0^\gamma) - x_2^\gamma(\gamma q)^{k_2} & \left(z_0\varphi(x_1^\gamma) - x_2^\gamma(\gamma z_0)\right) X_0 & \left(z_2\varphi(x_2^\gamma) - x_2^\gamma(\gamma z_2)\right) X_2 \end{pmatrix}$$

We want $q^{k_1}\varphi(x_1^\gamma) = x_1^\gamma(\gamma q)^{k_1}$, $q^{k_2}\varphi(x_0^\gamma) = x_2^\gamma(\gamma q)^{k_2}$, $x_0^\gamma = \varphi(x_2^\gamma)$ and $\left(z_0\varphi(x_1^\gamma) - x_2^\gamma(\gamma z_0)\right) X_0$, $\left(z_1\varphi(x_2^\gamma) - x_1^\gamma(\gamma z_1)\right) X_1$, $\left(z_2\varphi(x_2^\gamma) - x_2^\gamma(\gamma z_2)\right) X_2 \in \pi^k \mathcal{O}_E[[\pi, X_0, X_1, X_2]]$, where $k = k_1 + k_2$. By Lemma 2.3, $x_0^\gamma = \varphi(\lambda_{2,\gamma})^{k_2}$, $x_1^\gamma = (\lambda_{1,\gamma})^{k_1}$, $x_2^\gamma = (\lambda_{2,\gamma})^{k_2}$. We need to define z_1 so that $z_1\varphi(x_2^\gamma) - x_1^\gamma(\gamma z_1) \in \pi^k\mathbb{Z}_p[[\pi]]$ for all $\gamma \in \Gamma_K$. Since $x_0^\gamma = \varphi(y_1^\gamma)$ and $x_1^\gamma \in 1 + \pi^k\mathbb{Z}_p[[\pi]]$ (see Corollary 4.3 part (2)) it suffices to have $z_1 - \frac{B}{\gamma B}\gamma z_1 \in \pi^k\mathbb{Z}_p[[\pi]]$ for all $\gamma \in \Gamma_K$, where $B = \frac{\lambda_1^{k_1}}{\varphi(\lambda_2)^{k_2}}$. Since $B \in \mathcal{R}$ (see Corollary 4.4) and $B \equiv 1 \bmod \pi$, the existence of z_1 follows from Corollary 4.4. The existence of z_0 and z_2 are proven similarly and the existence of the matrix $G_\gamma^{(k)}$ such that $\Pi(\vec{X})\varphi(G_\gamma^{(k)}) - G_\gamma^{(k)}\gamma(\Pi(\vec{X})) \in \pi^k\mathcal{M}_3$ for all $\gamma \in \Gamma_K$, and $G_\gamma^{(k)} \equiv \vec{Id} \bmod \pi$ follows. Let $P = \Pi \bmod \pi$, then $(p^{k_1+k_2}P^{-1}) \bmod (p, X_0, X_1, X_2) = 0$, therefore the operator in condition (4) of Theorem 3.4 is the identity. A direct computation shows that if $HP = p^tPH$ for some $t > 0$, then $H = 0$, therefore Theorem 3.4 applies. For each $\gamma \in \Gamma_K$, there exists a unique matrix $G_\gamma(\vec{X}) \in \mathcal{M}_2$ such that $G_\gamma(\vec{X}) \equiv \vec{Id} \bmod \pi$ and $\Pi(\vec{X})\varphi(G_\gamma(\vec{X})) = G_\gamma(\vec{X})\gamma(\Pi(\vec{X}))$. For any $\vec{a} \in m_E^f$ the module $\boldsymbol{N}(\vec{a}) = (\mathcal{O}_E[[\pi]])\,\eta_1 \oplus (\mathcal{O}_E[[\pi]])\,\eta_2 \oplus (\mathcal{O}_E[[\pi]])\,\eta_3$ equipped with semilinear φ and Γ_K-actions defined by $(\varphi(\eta_1), \varphi(\eta_2), \varphi(\eta_3)) = (\eta_1, \eta_2, \eta_3)\Pi(\vec{a})$ and $(\gamma(\eta_1), \gamma(\eta_2), \gamma(\eta_3)) = (\eta_1, \eta_2, \eta_3)G_\gamma(\vec{a})$ is a Wach module corresponding to some lattice in a crystalline 3-dimensional E-representation of $G_{\mathbb{Q}_p}$.

Let $\vec{a} \in m_E^3$, since $z_i \equiv p^m \bmod \pi$, with $m = \lfloor \frac{k_1+k_2-1}{p-1} \rfloor$, we have $P(\vec{a}) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & p^{k_1} & \alpha_1 \\ p^{k_2} & \alpha_0 & \alpha_2 \end{pmatrix}$, where $\alpha_i = a_i p^m$. Arguing as in Proposition 4.15 one sees that

$$E \bigotimes_{\mathcal{O}_E} \left(\mathrm{Fil}^j(\boldsymbol{N}(\vec{a})/\pi \boldsymbol{N}(\vec{a}))\right) = \begin{cases} E\bar{\eta}_1 \oplus E\bar{\eta}_2 \oplus E\bar{\eta}_3 \text{ if } j \leq 0, \\ E\bar{\eta}_1 \oplus E\bar{\eta}_2 \text{ if } 1 \leq j \leq \min\{k_1, k_2\}, \\ E\bar{\eta}_t \text{ if } 1 + \min\{k_1, k_2\} \leq j \leq \max\{k_1, k_2\}, \\ 0 \text{ if } j \geq 1 + \max\{k_1, k_2\}, \end{cases}$$

where $\bar{\eta}_t = \bar{\eta}_1$ if $k_1 < k_2$ and $\bar{\eta}_t = \bar{\eta}_2$ if $k_2 < k_1$. For any $\vec{\alpha} \in p^m m_E^3$, we define families of filtered φ-modules $(D(\vec{\alpha}), \varphi(\vec{\alpha}))$ by setting $[\varphi(\vec{\alpha})]_{\underline{\eta}} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & p^{k_1} & \alpha_1 \\ p^{k_2} & \alpha_0 & \alpha_2 \end{pmatrix}$ and

$$Fi\ell^j D(\vec{\alpha}) = \begin{cases} E\eta_1 \oplus E\eta_2 \oplus E\eta_3 \text{ if } j \leq 0, \\ E\eta_1 \oplus E\eta_2 \text{ if } 1 \leq j \leq \min\{k_1, k_2\}, \\ E\eta_t \text{ if } 1 + \min\{k_1, k_2\} \leq j \leq \max\{k_1, k_2\}, \\ 0 \text{ if } j \geq 1 + \max\{k_1, k_2\}, \end{cases}$$

where $\eta_t = \eta_1$ if $k_1 < k_2$ and $\eta_t = \eta_2$ if $k_2 < k_1$.They are weakly admissible corresponding to 3-dimensional crystalline representations $V(\vec{\alpha})$ of $G_{\mathbb{Q}_p}$ with Hodge-Tate weights $\{0, -k_1, -k_2\}$.

Proposition 7.1 *The following isomorphisms hold:*
(i) For any $\vec{\alpha} \in p^m m_E^3$, $\bar{V}(\vec{\alpha}) \simeq \bar{V}(\vec{0})$;
(ii) If $p+1 \nmid k_2$, *then* $\bar{V}(\vec{0}) \simeq \omega^{-k_1} \oplus Ind_{G_{\mathbb{Q}_p}}^{G_{\mathbb{Q}_{p^2}}}(\omega_2^{-k_2})$;
(iii) If $p+1 \mid k_2$, *then* $\bar{V}(\vec{0}) \simeq \omega^{-k_1} \oplus \mu_{\sqrt{-1}} \omega^{-\frac{k_2}{p+1}} \oplus \mu_{-\sqrt{-1}} \omega^{-\frac{k_2}{p+1}}$,
where ω *is the* mod p *cyclotomic character of* $G_{\mathbb{Q}_p}$, $\mu_{\pm\sqrt{-1}}$ *the unramified character of* $G_{\mathbb{Q}_p}$ *which maps* $Frob_p$ *to* $\pm\sqrt{-1}$ *and* ω_2 *the fundamental character of level 2.*

Proof. Part (i) is immediate by Theorem 3.7. We change the base to have the matrix of Frobenius in diagonal form. Let $\varepsilon_1 = p^{k_1}$, $\varepsilon_2 = \sqrt{-1}p^{\frac{k_2}{2}}$ and $\varepsilon_3 = -\sqrt{-1}p^{\frac{k_2}{2}}$. If $A = \begin{pmatrix} 0 & 1 & 0 \\ -\varepsilon_2 & 0 & 1 \\ -\varepsilon_3 & 0 & 1 \end{pmatrix}$, then APA^{-1} =diag$(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, where $P = [\varphi(\vec{0})]_{\underline{\eta}}$. Let $\underline{\xi}$ be the ordered base of $D(\vec{0})$ defined by $\underline{\eta} = \underline{\xi} \cdot A$, then $[\varphi(\vec{0})]_{\underline{\xi}}$ =diag$(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and

$$Fi\ell^j D(\vec{0}) = \begin{cases} E\xi_1 \oplus E\xi_2 \oplus E\xi_3 \text{ if } j \leq 0, \\ E\xi_1 \oplus E(\xi_2 + \xi_3) \text{ if } 1 \leq j \leq \min\{k_1, k_2\}, \\ E\xi_t \text{ if } 1 + \min\{k_1, k_2\} \leq j \leq \max\{k_1, k_2\}, \\ 0 \text{ if } j \geq 1 + \max\{k_1, k_2\}, \end{cases}$$

where $\xi_t = \xi_2 + \xi_3$ if $k_1 < k_2$ and $\xi_t = \xi_1$ if $k_2 < k_1$. One sees that the submodules $E\xi_1$ and $E\xi_2 \oplus E\xi_3$ are φ-stable and it is trivial to show that $t_N^E(E\xi_2 \oplus E\xi_3) = k_2$. On the other hand,

$$Fi\ell^j\,(E\xi_2 \oplus E\xi_3) = \begin{cases} E\xi_2 \oplus E\xi_3 \text{ if } j \leq 0, \\ E(\xi_2 + \xi_3) \text{ if } 1 \leq j \leq \min\{k_1, k_2\}, \\ E\xi_t' \text{ if } 1 + \min\{k_1, k_2\} \leq j \leq \max\{k_1, k_2\}, \\ 0 \text{ if } j \geq 1 + \max\{k_1, k_2\}, \end{cases}$$

where $\xi_t' = \xi_2 + \xi_3$ if $k_1 < k_2$ and $\xi_t' = 0$ if $k_2 < k_1$. From this it follows that $t_H^E(E\xi_2 \oplus E\xi_3) = k_2$ (c.f. [12, § 2.4]) and $E\xi_2 \oplus E\xi_3$ is an admissible sub-object. Similarly, one proves that $E\xi_1$ is admissible. The filtered module $E\xi_1$ corresponds to χ^{-k_1}, where χ is the cyclotomic character of $G_{\mathbb{Q}_p}$. The filtered φ-submodule $E\xi_2 \oplus E\xi_3$ is the same as $D_{k_2+1,0}$ in [5, § 3] with Frobenius diagonalized. The Proposition follows from [5, Prop. 3.2], bearing in mind that we are dealing with the dual representation. ■

Remark 7.2 *In the three-dimensional, case other families of Wach modules of G_K can be constructed by using matrices $\Pi = \Pi_1 \times \Pi_2 \times ... \times \Pi_{f-1} \times \Pi_0$, with Π_i being matrices of the formed used above along with e.g. matrices of the type* $\begin{pmatrix} 0 & -1 & -1 \\ q^{w_{i1}} & X_{i0}z_0 & X_{i1}z_1 \\ 0 & 0 & q^{w_{i2}} \end{pmatrix}$. *The corresponding crystalline representations will have labeled weights $(\{0, -w_{i1}, -w_{i2}\})_{\tau_i}$. Analogous constructions apply in any dimension.*

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